μ-Limit Sets of Cellular Automata

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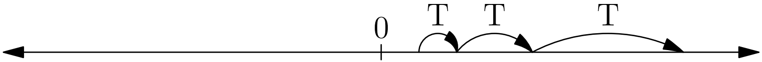
Journées SDA2 2015

General framework

Consider a dynamical system, for example $(\overline{\mathbb{R}}, T : x \mapsto 2x)$.



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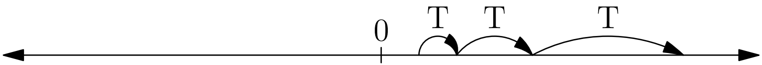

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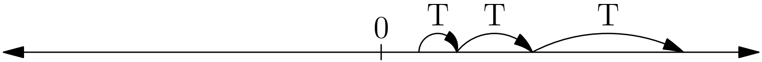

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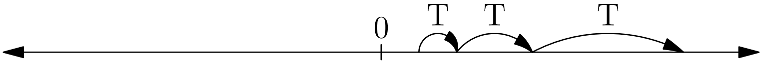

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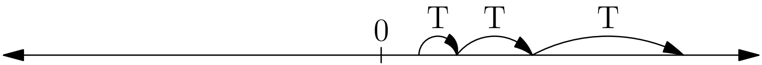

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$$\forall n \in \mathbb{N}, \mu(T^{-n}([a,b])) = \mu\left(\left[\frac{a}{2^n}, \frac{b}{2^n}\right]\right)$$

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As a dynamical system, a d-dimensional Cellular Automaton (CA) F is a shift-invariant continuous transformation of X. Equivalently, is is given by an alphabet $\mathcal{A}$, a finite neighborhood $\mathcal{N} \subset \mathbb{Z}^d$ and a local function $\delta : \mathcal{A}^{\mathcal{N}} \to \mathcal{A}$, such that:

$$\forall c \in X, \forall, s \in \mathbb{Z}^d, F(c)_s = \delta(c_{s+\mathcal{N}})$$

Example

As an example, take the MAX automaton, defined on alphabet $\mathcal{A} = \{0, 1\}$ and neighborhood $\mathcal{N} = \{(i_1, \dots, i_d), \sum_j |i_j| \leq 1\}$ by local rule

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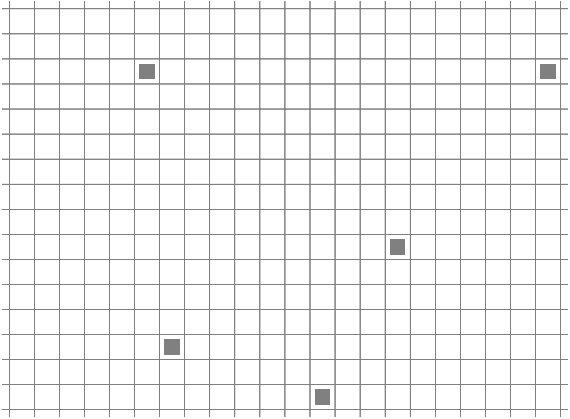

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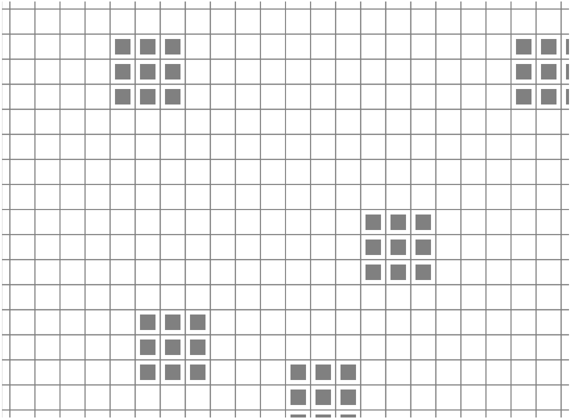

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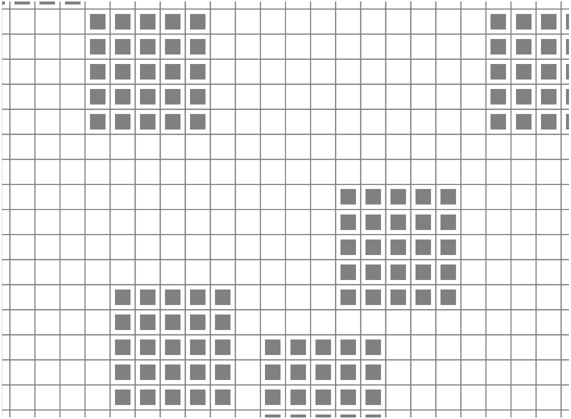

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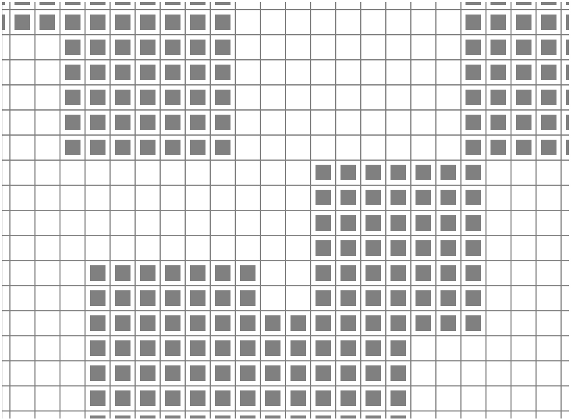

Limit set

Define the limit language as:

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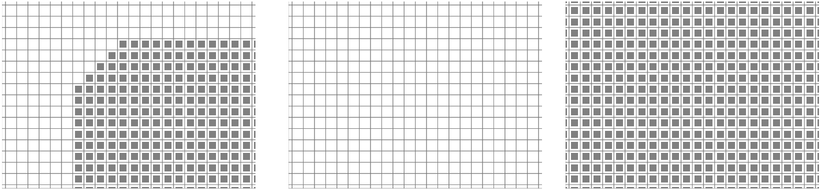
In the case of MAX, there are infinitely many configurations in $\Lambda(F)$.



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Define now the μ-limit language for some μ shift-invariant:

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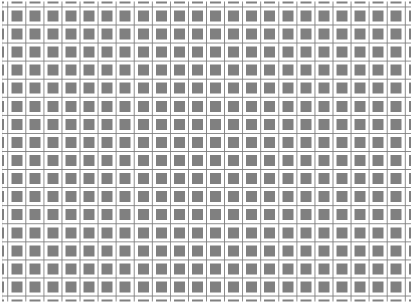
In the case of MAX, for every "reasonable" μ, the μ-limit set contains only the uniform configuration.



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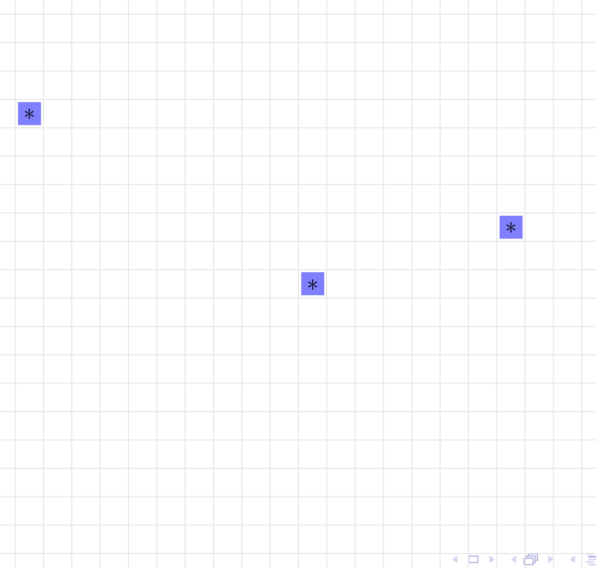
This is essentially a way to answer previous questions, in particular it gives computability results on μ-limit sets, and can be used to construct "interesting" ones.

Construction: cleaning the space

One of the principles is to clean the space using one special state: the *seed*.



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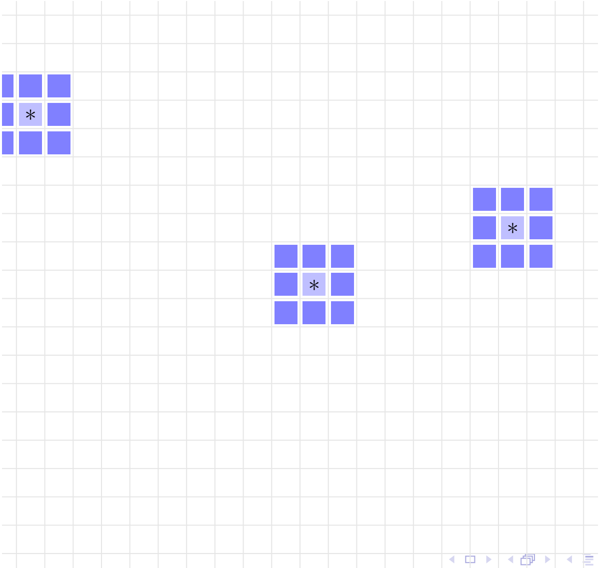

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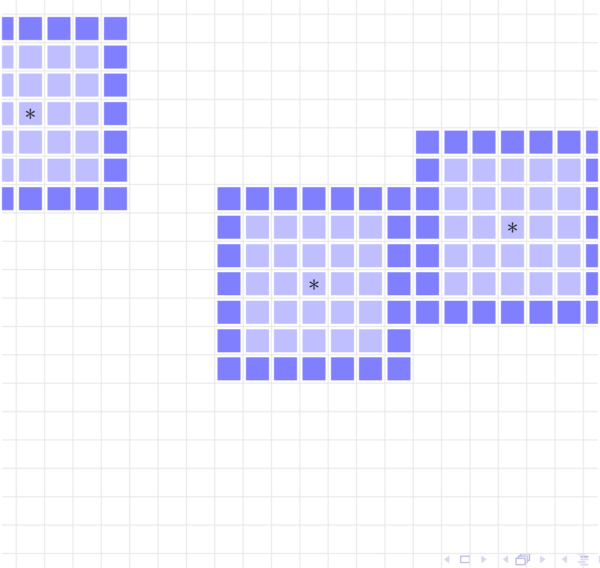

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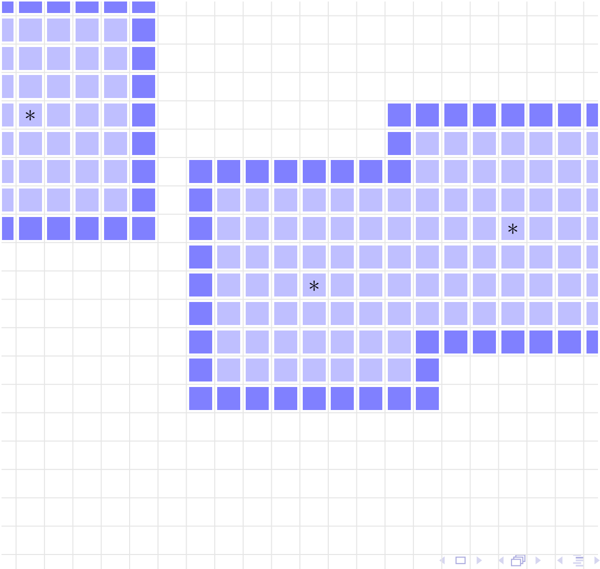

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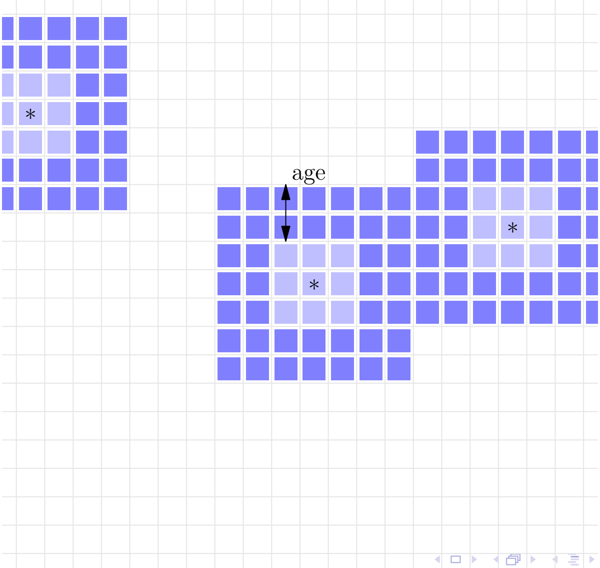

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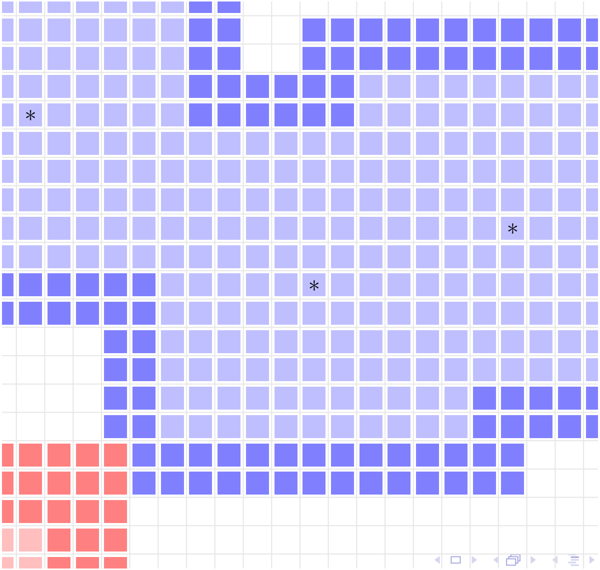

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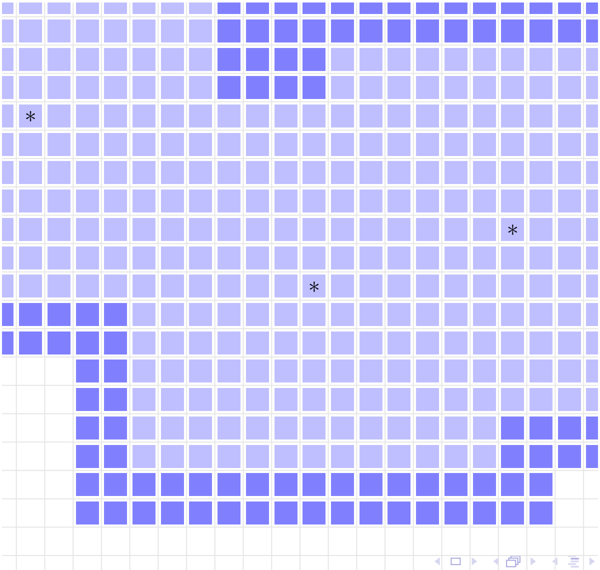
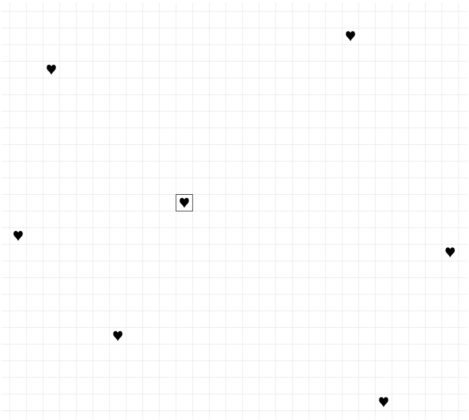


Construction: struggle for life

Once the space is cleaned, seeds become hearts, and they need more and more space to live as centers of organisms. At each time $t_n = 2^n$, a new period starts and organisms grow as much as they can.



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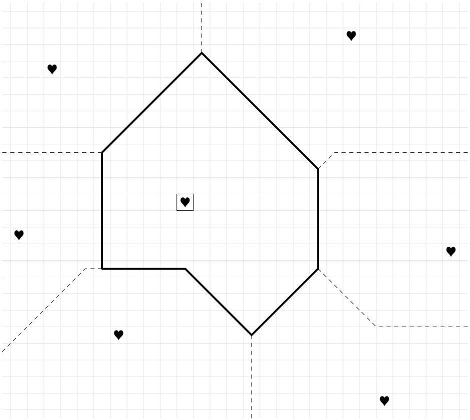

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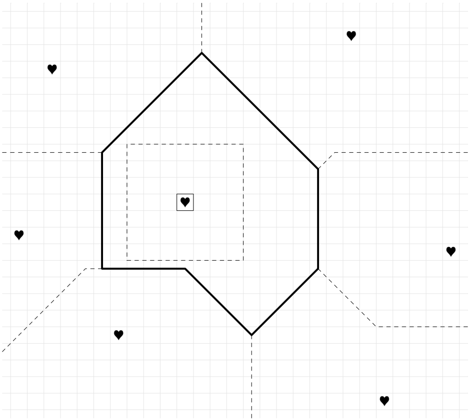

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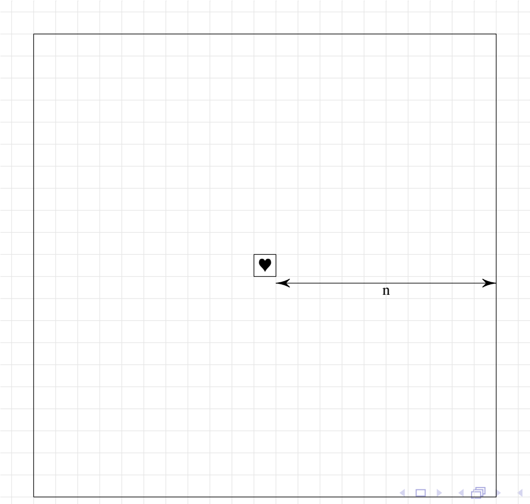

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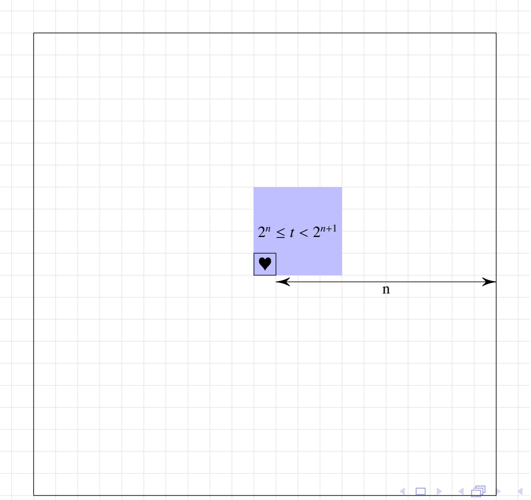
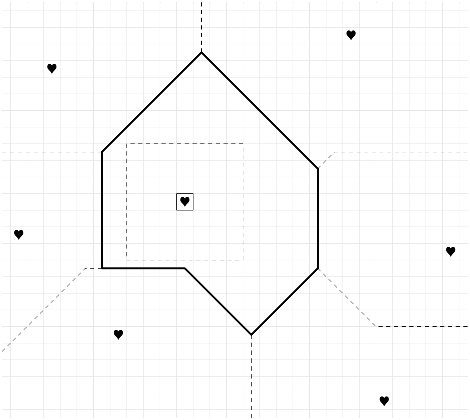


Construction: cycle of life

During each period, each organism computes some w_i and writes it all over its space.



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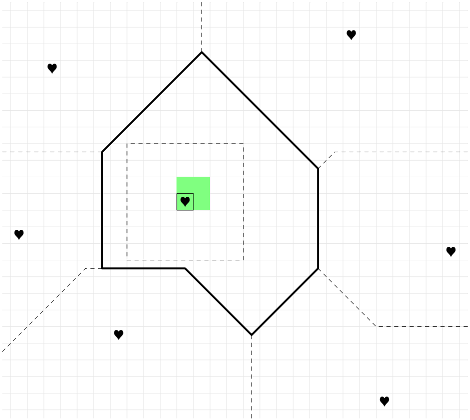

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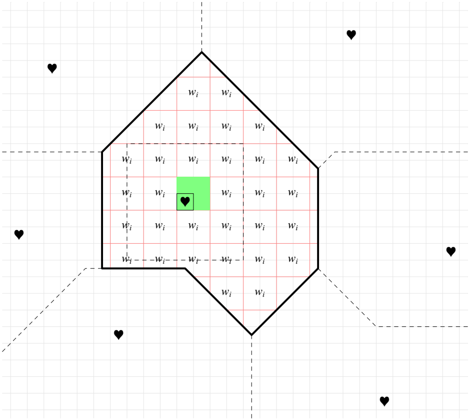

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Also, we have a Rice theorem on μ-limit sets:

Theorem

Any non-trivial property of μ-limit sets is at least Π_3-hard.

Future work

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