On automorphism groups of low complexity minimal subshifts

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A topological dynamical system is a pair (X, T) where X is a compact metric space and $T\colon X \to X$ is a homeomorphism.

In this talk we assume that (X, T) is a minimal system: any orbit $\{T^n x\colon n \in \mathbb{Z}\}$ is dense in X. This is equivalent to: the only closed invariant subsets are $\emptyset$ and X.

An automorphism $\phi\colon X \to X$ is an homeomorphism s.t.

$$\phi \circ T = T \circ \phi.$$

$$\mathrm{Aut}(X, T) = \{\phi \text{ automorphism of } (X, T)\}.$$

$$\{T^n : n \in \mathbb{Z}\} \subset \mathrm{Aut}(X, T).$$

Moreover, $\{T^n : n \in \mathbb{Z}\}$ is included in the center of $\mathrm{Aut}(X, T)$.

Let $\mathcal{A}$ be a finite alphabet. Let $X \subset \mathcal{A}^{\mathbb{Z}}$ be a subshift: closed and invariant by the shift action

$$\begin{array}{rcl} \sigma : X & \to & X \\ (x_n)_{n\in\mathbb{Z}} & \mapsto & (x_{n+1})_{n\in\mathbb{Z}} \end{array}$$

Question

What can be said about $\mathrm{Aut}(X, \sigma)$?

Theorem (Curtis-Hedlund-Lyndon)

Let ϕ *be an automorphism of* (X, σ)
There exists a local map $\hat{\phi}$: $\mathcal{A}^{2r+1} \to \mathcal{A}$ *s.t.*

$$\phi(x)_n = \hat{\phi}(x_{n-r} \dots x_{n+r}) \text{ for any } n \in \mathbb{Z}.$$

Then $\mathrm{Aut}(X, \sigma)$ is a countable group.

Centralizer group: for a measurable dynamical system $(X, \mathcal{B}, \mu, T)$,

$$C(T) = \{\phi\colon X \to X;\ \text{bi-measurable},\ \phi\mu = \mu \text{ and } \phi \circ T = T \circ \phi\}$$

Previous results in the measurable setting

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J. King, J.-P. Thouvenot (91): for mixing systems of finite rank

$$C(T)/\langle T \rangle \text{ is a finite group.}$$

G. A. Hedlund (69) : $\mathrm{Aut}(\mathcal{A}^{\mathbb{Z}}, \sigma)$ contains many subgroups:

- countable generated free groups,
- free product of cyclic groups,
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M. Hochman (2010) : Higher dimensional SFTs with positive entropy admit any finite group in $\mathrm{Aut}(X, T)$.

From the measurable to the topological setting

In the previous examples, the shifts have positive topological entropy.

For zero-entropy system,

B. Host, F. Parreau (89) : for a family of substitutive systems

$$C(\sigma) = \mathrm{Aut}(X, \sigma) \text{ and } C(\sigma)/\langle \sigma \rangle \text{ is a finite group .}$$

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M. Lemańczyk, M. Mentzen (89): any finite group can be realized as $C(\sigma)/\langle\sigma\rangle$.

The complexity function $p_X : \mathbb{N} \to \mathbb{N}$,

$$p_X(n) = \# \text{ words of length } n \text{ in the language of } X.$$

Recall that

$$h_{top}(X, \sigma) = \lim \frac{log(p_X(n))}{n}$$

Question

What can be said for systems with low complexity function?

We can ask for sublinear complexity, or more generally sublinear complexity along a subsequence:

Theorem (DDMP, 2014)

Let (X, σ) be a minimal subshift. If

$$\liminf_n \frac{p_X(n)}{n} < +\infty,$$

then $Aut(X, \sigma)/\langle\sigma\rangle$ is a finite group.

Includes primitive substitutions, linear recurrent systems, interval exchange transformations.

Answers a question of V. Salo and I. Törmä, who proved the same result for Pisot or constant length substitutions.

Also discovered by V. Cyr and B. Kra and E. Coven and R. Yassawi.

Main Ideas

The main idea we use is to study the action of $\mathrm{Aut}(X,\sigma)$ on special pairs.

Lemma

Let (X,T) be a minimal aperiodic dynamical system. The action of $\mathrm{Aut}(X,T)$ on X

$$\begin{array}{rcl} \mathrm{Aut}(X,T)\times X & \to & X \\ (\phi,x) & \mapsto & \phi(x), \end{array}$$

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Proof. For any automorphism ϕ, the set

$$\{x : \phi(x) = x\}$$

is closed and T invariant.

Main Ideas

Two points $x, y \in X$ are asymptotic if

$$\lim_{n \to +\infty} \mathrm{d}(T^n(x), T^n(y)) = 0.$$

An infinite subshift always admits an asymptotic pair.

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$$\lim_{n \to +\infty} d(T^n \phi(x), T^n \phi(y)) = \lim_{n \to +\infty} d(\phi T^n(x), \phi T^n(y)) = 0.$$

ϕ induces a permutation on the collection of asymptotic pair.

Main Ideas

$\{x, y\} \sim \{x', y'\}$ if x and x' are in the same T-orbit.
(*i.e.* $(x', y') = (T^n x, T^n y)$).

$\mathcal{AS}$ denote the collection of asymptotic unordered pairs.

$\mathcal{AS}_{/\sim}$ denotes the set of equivalence classes.

$\mathrm{Aut}(X, T)/\langle T \rangle$ acts on $\mathcal{AS}_{/\sim}$ (a permutation)

Proposition

Let (X, σ) *be a subshift with*

$$\liminf_n p_X(n)/n < \infty.$$

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Proof: $K = \liminf_n p_X(n)/n$.
claim: $p_X(n+1) - p_X(n) \leq \lfloor K \rfloor$ infinitely often. By contradiction: $\forall n \geq m$ big enough

$$\begin{aligned} p_X(n) - p_X(m) &= \sum_{i=m}^{n-1} p_X(i+1) - p_X(i) \geq (n-m)(\lfloor K \rfloor + 1) \\ p_X(n) &\geq (n-m)(\lfloor K \rfloor + 1) + p_X(m) \end{aligned}$$

Automorphism and factor maps.

We also study $\mathrm{Aut}(X, \sigma)$ through factor maps.

A factor map π is a continuous onto function

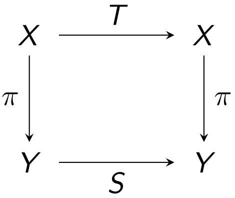

Figure : The diagram commutes.

A factor map $\pi\colon (X,T) \to (Y,S)$ is compatible with $\mathrm{Aut}(X,T)$ if

$$\pi(x) = \pi(x') \Longleftrightarrow \pi(\phi(x)) = \pi(\phi(x')) \text{ for every } \phi \in \mathrm{Aut}(X,T)$$

We can define $\widehat{\pi}\colon \mathrm{Aut}(X,T) \to \mathrm{Aut}(Y,S)$.

$$\widehat{\pi}(\phi)\pi(x) = \pi(\phi x).$$

$x, y \in X$ are proximal if

$$\liminf_n d(T^n x, T^n y) = 0.$$

i.e.

$$\lim_{i\to\infty} d(T^{n_i} x, T^{n_i} y) = 0 \text{ for some sequence } (n_i)_{i\in\mathbb{N}}.$$

A factor map $\pi\colon (X, T) \to (Y, S)$ is proximal if $\pi(x) = \pi(x')$ implies x, x' proximal.

A class of compatible systems

Commutator in a group G: $[g,h] = ghg^{-1}h^{-1}$
Commutator subgroups:

$$G_1 = G, \qquad G_j = [G_{j-1}, G] = \langle [a,b];\; a \in G_{j-1}, b \in G \rangle.$$

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A group G is d-step nilpotent if $G_{d+1} = \{e\}$.

Example. If $d = 1$, G is abelian.

G a d-step nilpotent Lie group. $\Gamma \subset G$ a subgroup cocompact. The homogeneous space G/Γ is a d-step nilmanifold .

G/Γ endowed with a minimal translation $L_g : h\Gamma \to gh\Gamma$ in G/Γ is a d-step nilsystem.

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Theorem (DDMP, 2014)

If $\pi \colon (X, T) \to (G/\Gamma, L_g)$ is a proximal extension of a minimal d-step nil system, then $Aut(X, T)$ is a d-step nilpotent group. Moreover, $\hat{\pi} \colon Aut(X, T) \to Aut(G/\Gamma, L_g)$ is injective.

Theorem (DDMP)

If (X, T) is a minimal proximal extension of its maximal non trivial d-step nilfactor (X_d, T_d). Then $\mathrm{Aut}(X, T)$ embeds into $\mathrm{Aut}(X_d, T_d)$, and $\mathrm{Aut}(X, T)$ is a d-step nilpotent group.

Example. Toeplitz subshifts are proximal extension of their maximal equicontinuous factor ($d = 1$). Their automorphism group is abelian.

Open questions

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Given a countable group G. Does it exists a minimal subshift such that $\mathrm{Aut}(X, \sigma)/\langle \sigma \rangle$ is isomorphic to G ?

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True for G finite, $\mathbb{Z}^d$.

V. Salo (2015): example $\mathrm{Aut}(X,\sigma)$ is abelian not finitely generated.

Realization of finite groups

$G = \{g_0, g_1, \ldots, g_n\}$. Consider the substitution (constant length)

$$\tau : g_i \to (g_i g_0)(g_i g_1) \cdots (g_i g_n)$$

Proposition (DDMP, 2014)

$\mathrm{Aut}(X_\tau, \sigma)/\langle\sigma\rangle$ *is isomorphic to* G*.*

Examples:

- Thue-Morse system, $0 \to 01, 1 \to 10$. $\mathrm{Aut}(X_\tau, \sigma)/\langle\sigma\rangle = \mathbb{Z}_2$.
- $0 \to 012$, $1 \to 120$, $2 \to 201$. $\mathrm{Aut}(X_\tau, \sigma)/\langle\sigma\rangle = \mathbb{Z}_3$.

Question

What can be said for subshifts with subpolynomial complexity?

Cyr and Kra (2015): if $\liminf p_X(n)/n^2 \to 0$ then $\mathrm{Aut}(X, \sigma)$ is periodic

Question (88)

Are $\mathrm{Aut}(\{0,1\}^{\mathbb{Z}}, \sigma)$ and $\mathrm{Aut}(\{0,1,2\}^{\mathbb{Z}}, \sigma)$ isomorphic?

Merci.