Deciding conjugacy for certain class of one-sided finite-type-Dyck shifts

Pavel Heller, joint work with Marie-Pierre Béal

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Deciding shift conjugacy: generally open problem

- decidable for 1-sided SFT's [Williams, 73]
- unknown for 1-sided sofic shifts
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Outline

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Shifts

- A finite alphabet
- A^* set of finite words over A
- $A^{\mathbb{Z}}$ set of bi-infinite sequences over A
- $A^{-\mathbb{N}}$ set left-infinite sequences over A

Shift

$X \subset A^{\mathbb{Z}}$ (resp. $A^{-\mathbb{N}}$) is called a (*one-sided*) *shift* if it is the set of sequences avoiding some set of forbidden words $F \subset A^*$. We denote $X = \mathsf{X}_F$.

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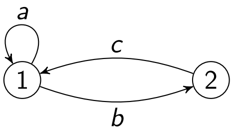
Example: shift of finite type

Let $A = \{a, b, c\}$. Consider X_F for $F = \{ba, bb, ac, cc\}$. Represented by the following local automaton.



Shift conjugacy

Block map

Given A, B alphabets and X shift over A. $\Phi : A^{\mathbb{Z}} \to B^{\mathbb{Z}}$ is a *block map* if there are non-negative integers m, a and a function $\phi : A^{m+1+a} \to B$ such that for any $x \in X$ and any i

$$\Phi(x)_i = \phi(x_{i-m} \cdots x_{i+a})\,.$$

Analougously can be defined for one-sided shifts, setting $a = 0$.

Conjugacy

Two shifts X, Y are *conjugate* if there is a bijective block map $\Phi : X \to Y$.

Note: the inverse is also block map.

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Dyck shifts consist of well-formed sequences of brackets.

E.g.

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Tripartitioned alphabet

Let the alphabet be divided into three disjoint sets of *call*, *return*, and *internal* symbols:

$$A = A_c \sqcup A_r \sqcup A_i \,.$$

Example

Consider the SFT defined by the automaton, where $A_c = \{\,(\,,[\,\}$, $A_r = \{\,),]\,\}$, $A_d = \{j,k\}$

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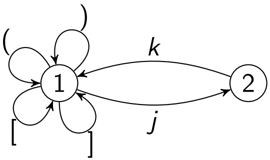

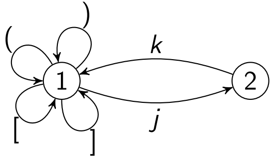

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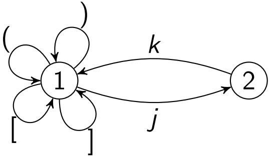
The SFT already avoids some finite set of forbiden factors F. So why not also forbid matching the pairs from the set $U = \{ \langle (,] \rangle , \langle [,) \rangle \}$.



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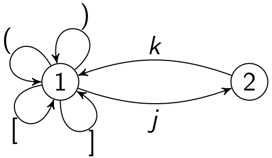

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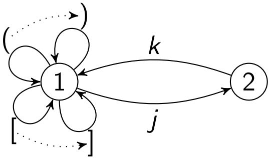

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The SFT already avoids some finite set of forbiden factors F. So why not also forbid matching the pairs from the set $U = \{ \langle (,] \rangle , \langle [,) \rangle \}$. We can record the same information in the automaton by joining edges whose symbols are allowed to match.

But we could (dis)allow any other combination of call and return symbols. Consider

corresponding to forbidden matching $U = \{ \langle (,] \rangle \}$.

We can also consider context. For example we may allow brackets matching only if at least one of them is preceded by k. Hence distinguishing

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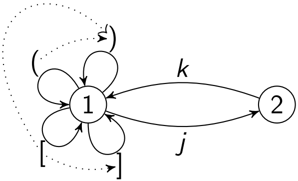

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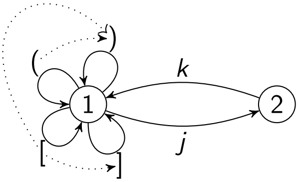

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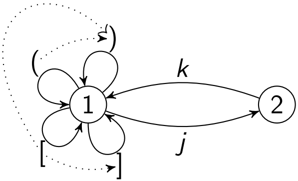

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1-sided finite-type-Dyck shift

We call *1-sided finite-type-Dyck shift* over alphabet A any set $X \subseteq A^{-\mathbb{N}}$ that avoids some finite set of forbidden words $F \subseteq A^{m+1}$ and finite set of matching patterns $U \subseteq A^m A_c \times A^m A_r$ for some non-negative integer m. We denote $X = \mathsf{X}_{F,U}$.

Some notes:

- Equivalently, 1-sided FTD shifts can be defined as left-infinite paths admissible in local automata with arbitrary matching relations between call and return edges. So-called Dyck automata.
- By construction, both (1-sided) Dyck shifts and SFT's are included.
- The language of factors is a visibly pushdown language.

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Outline

Proper block map

A block map Φ is *proper* when $Phi(x)_j \in A_c$ (resp. A_r, A_i) if and only if $x_j \in A_c$ (resp. A_r, A_i).

Matched-return word

A word over tripartitioned alphabet A is *matched-return* if it "has no unmatched return symbols". Formally, if each its prefix contains at least as many call symbols as return symbols.

MR-extensible shift

A shift X is called *MR-extensible* if for any of its blocks, u exists a non-empty matched-return word v such that uv is a block of X.

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The result

Theorem

It is decidable in an effective way whether two one-sided finite-type-Dyck shifts which are MR-extensible are properly conjugate.

- Not all 1-sided FTD shifts are MR-extensible, but many are.
- Dyck and Motzkin shifts are MR-extensible.
- Our examples were MR-extensible.

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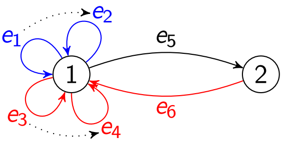
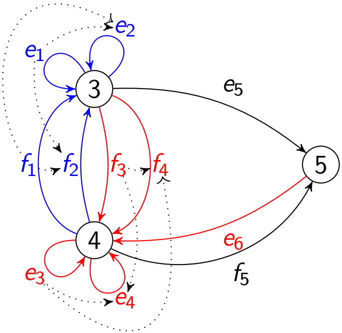
Deciding the conjugacy

In the process of deciding, the operations of *in-splitting* and *in-merging* Dyck graphs (multigraph with matching of some edges) are crucial.





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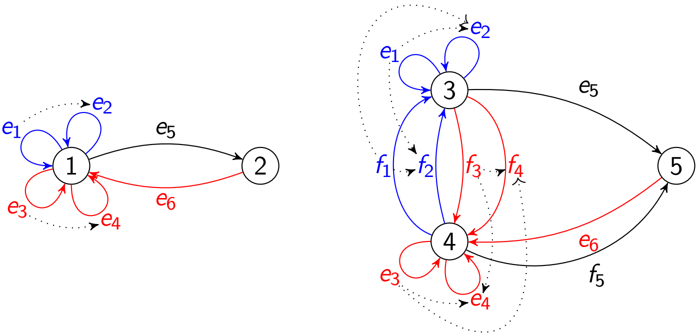

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Each new state receives a precise copy of its out-going edges (including the matching relations.

The original in-coming edges are partitioned between the two new states.

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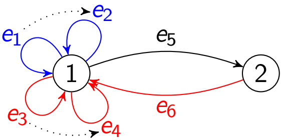

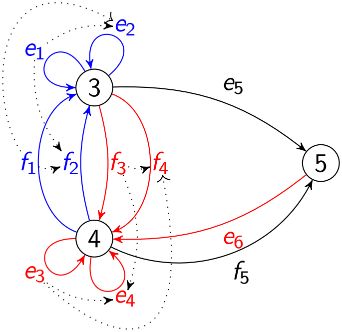

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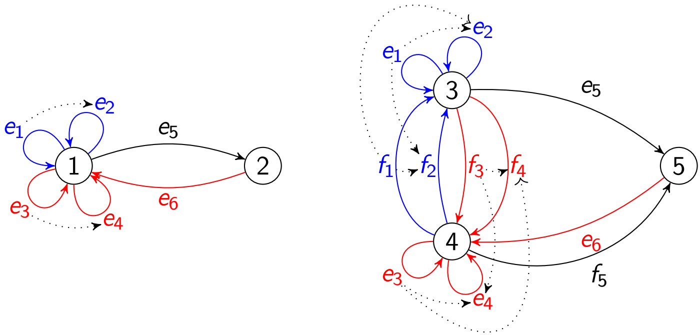

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Deciding the conjugacy

To decide conjugacy of two 1-sided FTD'S given by finite sets $F \subseteq A^{m+1}, U \subseteq A^m A_c \times A^m A_r$:

- construct from F, U, automaton of certain form recognizing the shift
- from now on disregard labels, work only with the underlying Dyck graph
- perform in-merges in the two graphs as long as possible
- check, whether they are isomorphic (in the Dyck graph sense, i.e. respecting the edge matching)

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