Aperiodic Tilings on Groups

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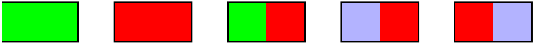
Goal of the talk

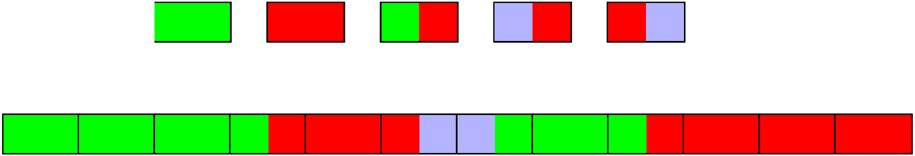
Introduce tilings (SFTs) on groups

Find computational obstructions to the existence of aperiodic tilings

Tilings on $\mathbb{Z}$

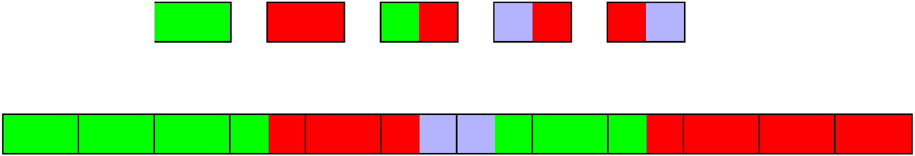


Tilings on $\mathbb{Z}$



- Related to finite automata theory
- If there is a tiling, there is a periodic one

Tilings on $\mathbb{Z}$



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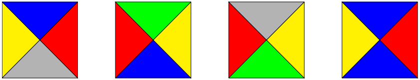
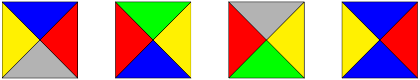
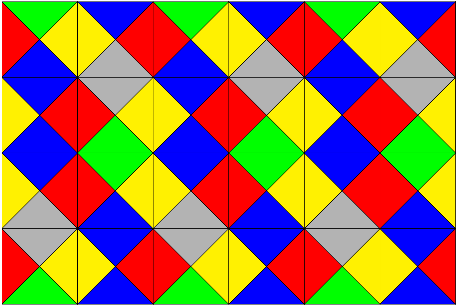
Tilings on $\mathbb{Z}^2$



Tilings on $\mathbb{Z}^2$





Tilings on a group

How to define tilings on a group ?

How to define periodicity on a group ?

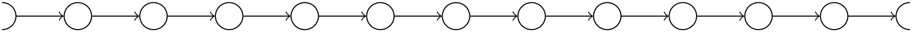
Cayley graph

Cayley graph of a group G with generators a, b

- Vertices are elements of G
- Edge from g to ga labelled by a.
- Edge from g to gb labelled by b.

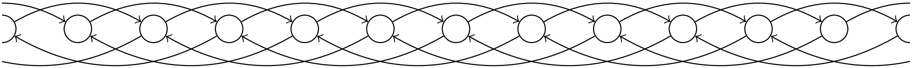
Cayley graph for $\mathbb{Z}$

Cayley graph for $\mathbb{Z}$, with $a = 1$



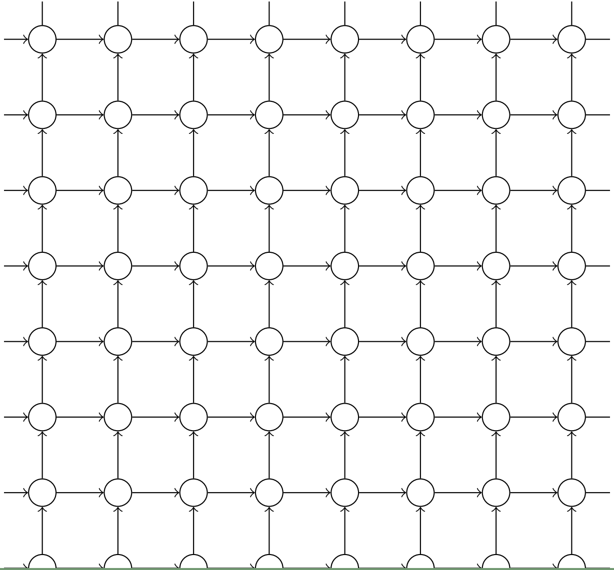
Cayley graph for $\mathbb{Z}$

Cayley graph for $\mathbb{Z}$, with $a = 2, b = -3$



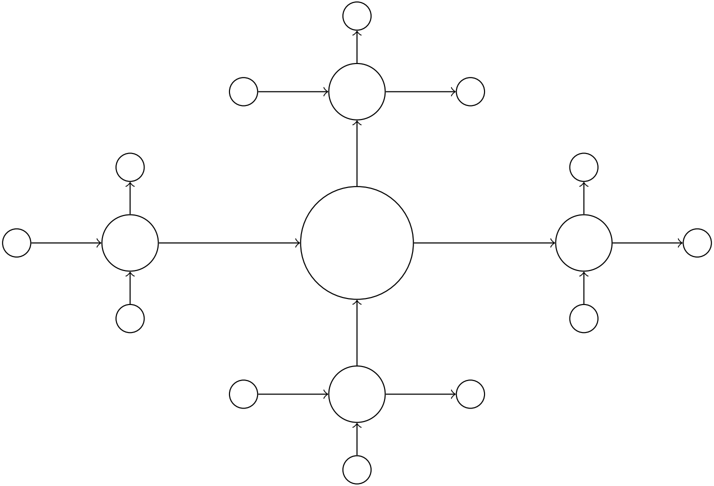
Cayley graph for $\mathbb{Z}^2$

$a = (1, 0)$, $b = (0, 1)$.



Cayley graph for the free group

The free group $\mathbb{F}_2$ with two generators a and b:

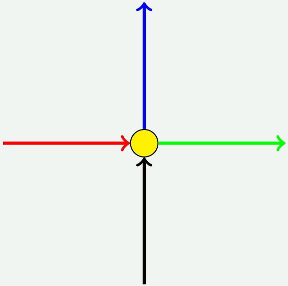


The free group

- All words on the alphabet $\{a, b, a^{-1}, b^{-1}\}$ with no factors $aa^{-1}, bb^{-1}, b^{-1}b, a^{-1}a$
- Multiplication is concatenation then simplification.

Tilings on groups

A Wang tile on a group with two generators:



Tilings on groups

Given a set of Wang tiles, is there a tiling of the group ?

Given a set of Wang tiles, is there a periodic tiling ?

Periodicity

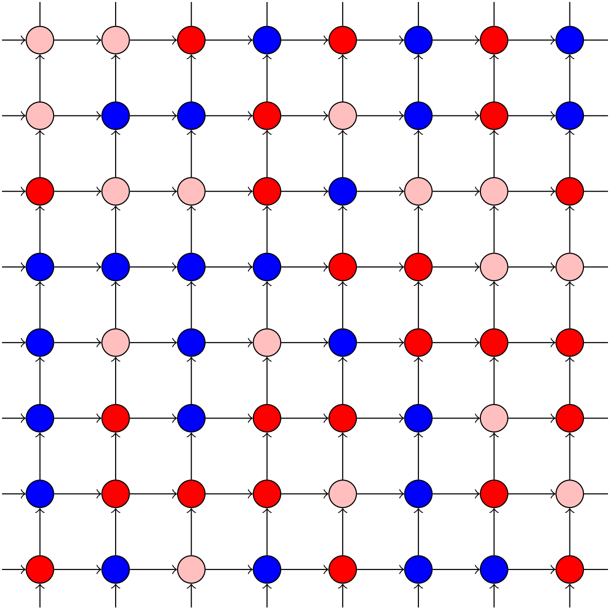
A tiling is a map $x : G \to W$ where W is the set of Wang tiles.

The translation of x by g is gx defined by

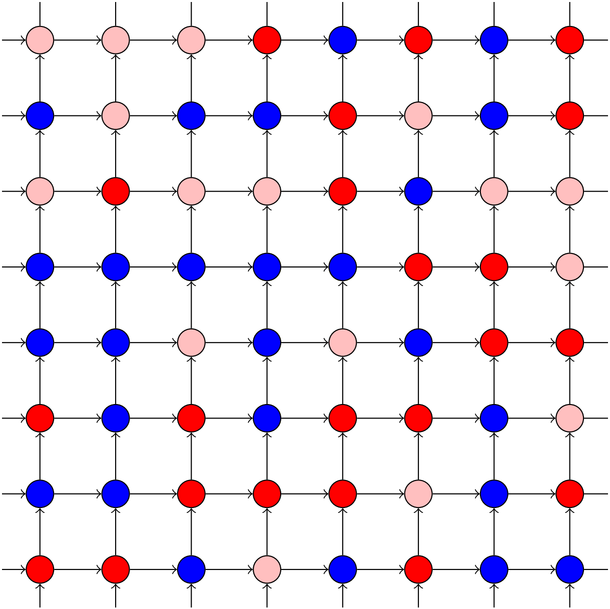
$$(gx)_h = x_{g^{-1}h}$$

x is periodic of period g if $gx = x$.

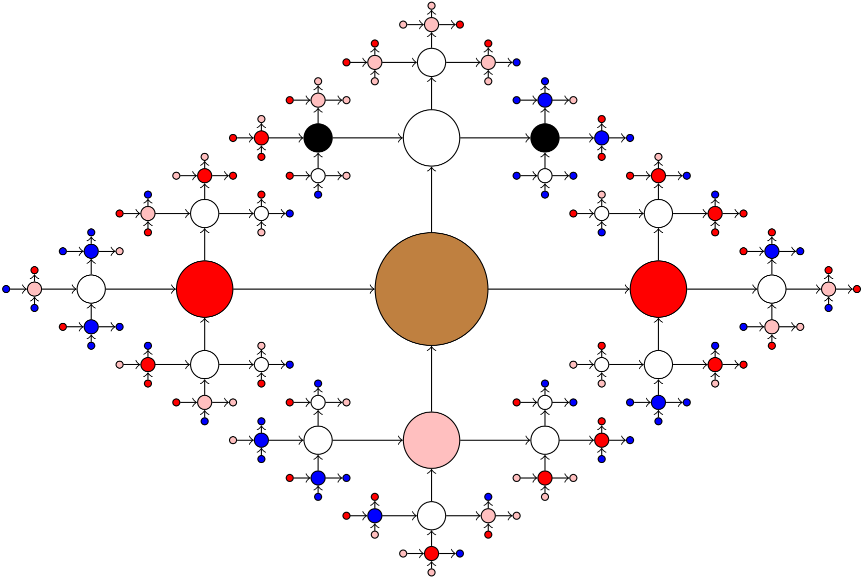
Example



Example



Example (free group)



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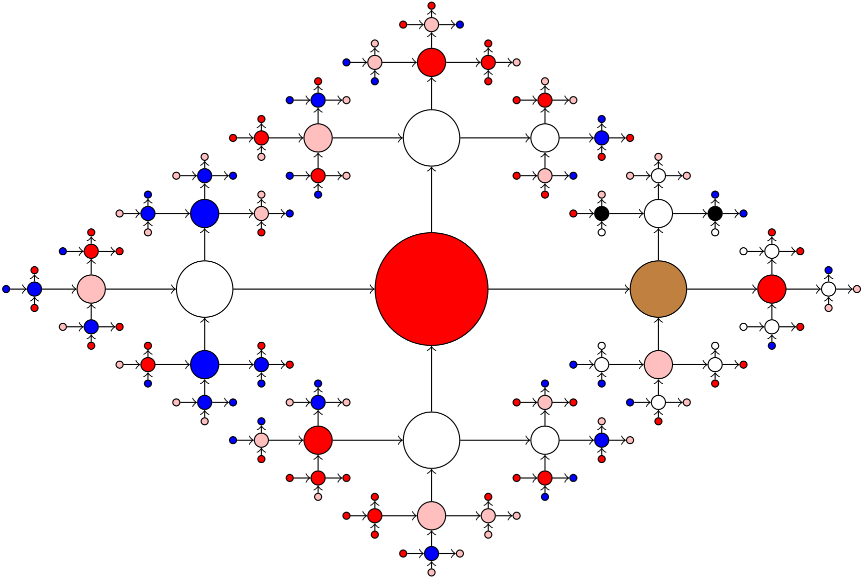
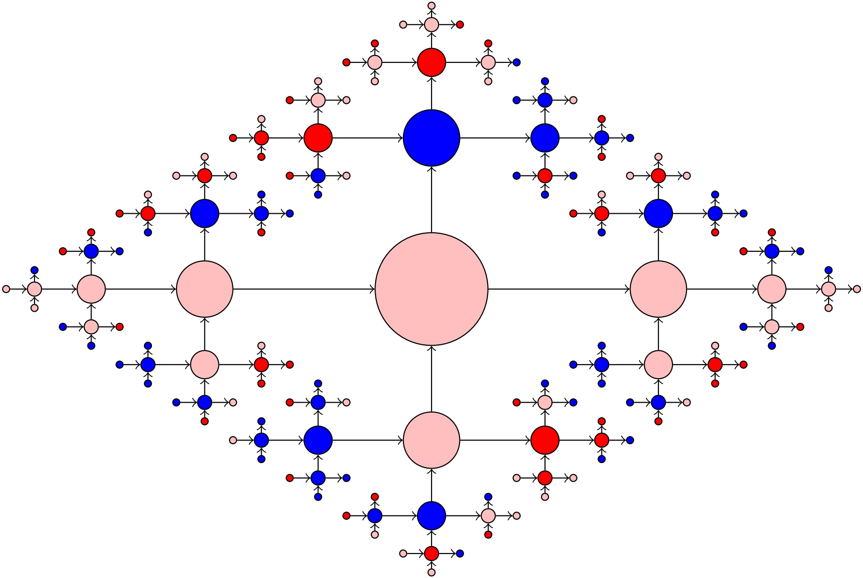
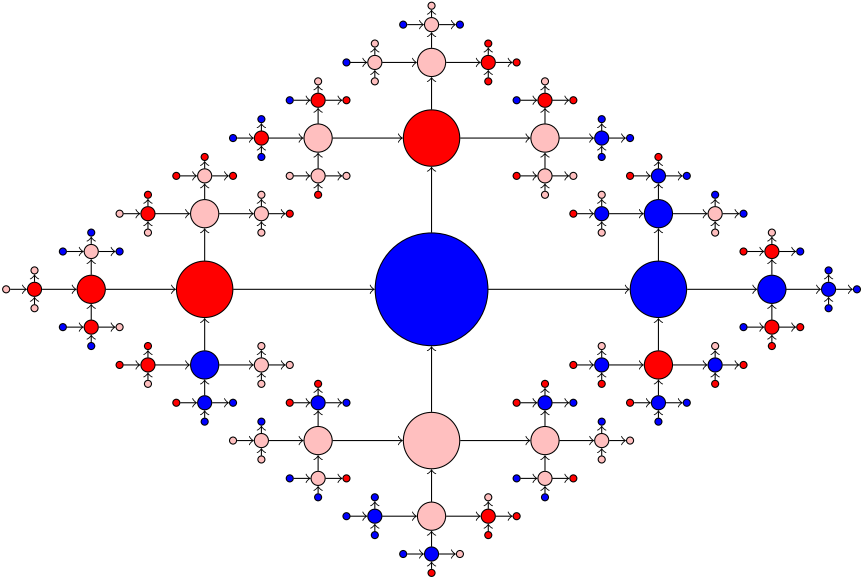


Periodic tiling (free group)



Periodic tiling (free group)



Beware

In noncommutative groups, counterintuitive things happen

If x is of period g, then hx is of period hgh^{-1}

Aperiodic tilings

Definition

A set of Wang tiles W is aperiodic if no tiling by W has a (nonzero) period.

(Not the classical definition, but easier for the talk)

On which group is it possible to have an aperiodic tileset ?

Previous work

- Aperiodic tiling on $\mathbb{Z}^2$ (Berger 1964)
- Aperiodic tiling on Baumslag-Solitar groups (Aubrun-Kari 2013)
- Aperiodic tiling on the Heisenberg Group (Sahin-Schraudner 2014)
- Aperiodic tiling implies the group is one ended (Cohen 2014)

First theorem

Theorem

If a f.p. group admits an aperiodic tileset, it has a decidable word problem

Word problem

Let G be generated by a and b. The *word problem* is to decide, given a word over $\{a, b, a^{-1}, b^{-1}\}$, whether it corresponds to the identity element on G

A group is *finitely presented* if there exists a finite set of relations R so that G is the largest group satisfying the set of relations.

$\mathbb{Z}^2 = \langle a, b | ab = ba \rangle$ $\mathbb{Z} = \langle a, b | b = 1 \rangle$ $\mathbb{F}_2 = \langle a, b | \rangle$

Idea of the proof

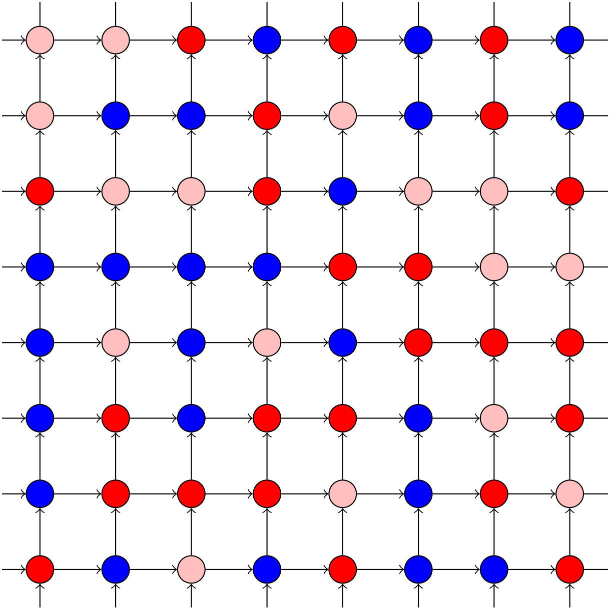
- Let G be a group generated by a and b
- Let F be the free group.
- The free group has a universal property: the map ϕ from F to G that maps a to a and b to b is a morphism.

Intuitively, the free group is the largest group with two generators. All other groups have additional relations

Idea of the proof

- Any tiling by W on G is a tiling by W on $\mathbb{F}_2$.
- We can "unfold" the Cayley graph/"unfold" the tiling.

Idea of the proof



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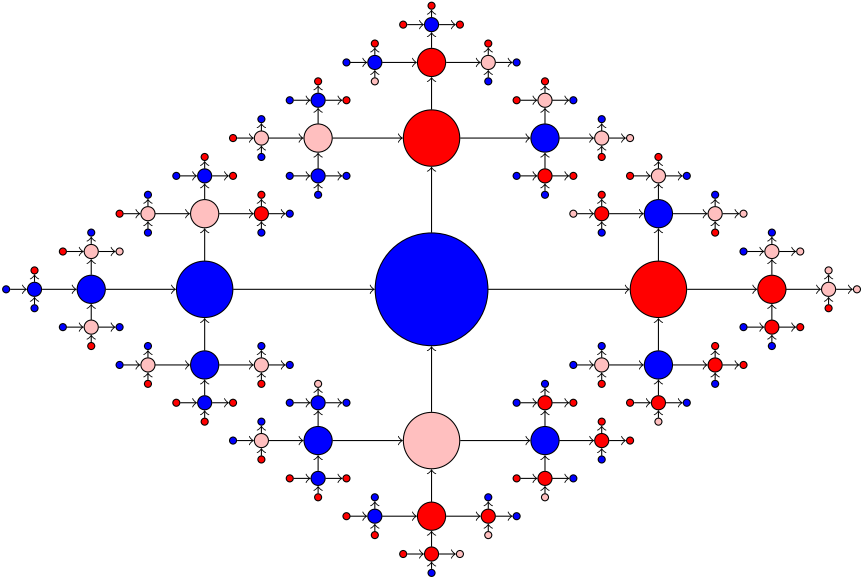


Idea of the proof

- Any tiling by W on G is a tiling by W on $\mathbb{F}_2$.
- Any tiling by W on $\mathbb{F}_2$ s.t.:

  $$\phi(g) = \phi(h) \implies x_h = x_g$$

  is a tiling on G.
  (ϕ is the morphism from $\mathbb{F}_2$ to G)
- If G is finitely presented there is an algorithm that can semidecide whether $\phi(g) = \phi(h)$

Idea of the proof

- Start from an aperiodic set W on G
- We want to decide the word problem. Equivalently, to decide, given $g \in \mathbb{F}_2$, if $\phi(g) = 1$.
- We can semidecide if $\phi(g) = 1$, so we should find an algorithm for $\phi(g) \neq 1$.
- Try to find a tiling x of $\mathbb{F}_2$ with the constraint that $x_{g^{-1}h} = x_h$ for all g.
  - If $\phi(g) = 1$ such a tiling exists
  - If $\phi(g) \neq 1$ there is no such tiling by aperiodicity
- This give a semidecision algorithm for $\phi(g) \neq 1$

First theorem - Enumeration version

Theorem

If a f.p. group admits an aperiodic tileset, it has a decidable word problem

Theorem

If a group admits an aperiodic tileset, then we can enumerate the complement of the word problem from the word problem

See my other talk at Paris-Est.

Other results

Try to find aperiodic tilesets for specific groups

The Kari tileset

Generalize a construction of Kari of an aperiodic tileset in $\mathbb{Z}^2$.

The tileset simulates a piecewise affine map $f : [0,1] \to [0,1]$:

- Every row corresponds to a real $x \in [0,1]$
- If row i corresponds to x, row $i+1$ corresponds to $f(x)$.
- Aperiodicity comes from the fact that f has no periodic points.

Generalizations

Let G be a group.
Start from two piecewise affine maps f_a and f_b s.t.:

$$f_w(x) = x \iff w = 1$$

or

$$\forall g, f_{gw}(x) = f_g(x) \iff w = 1$$

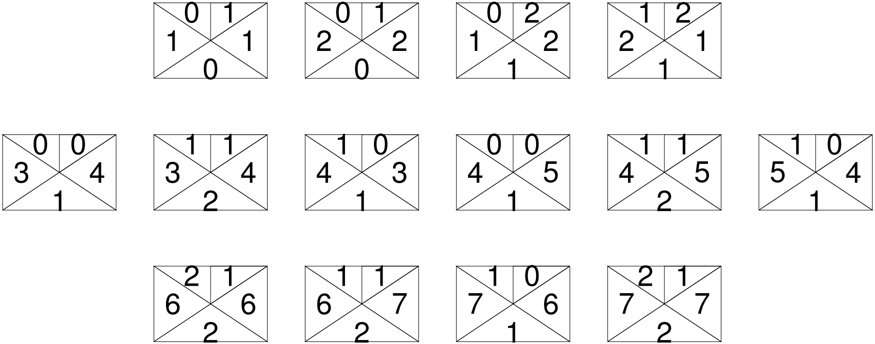
Then the construction gives an aperiodic set of tiles for $G \times \mathbb{Z}$

Second theorem

Theorem

For a large class of groups G, closed under product, and that contains free groups, Thompson's groups T and V, and compact matrix groups $G \times \mathbb{Z}$ admits an aperiodic tileset.

Example - $PSL_2(\mathbb{Z})$



(Colors on a^{-1} and b^{-1} are identical and represented only once)

Open question

- Prove that every group admits an aperiodic subshift.
- Generalize Kari construction in the other direction.