Géométrie fractale, algorithmes, calculabilité

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En collaboration avec **Jarkko Kari**

Journées SDA2
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Iterated function systems (IFS)

Let $f_1, \ldots, f_n : \mathbb{R}^d \to \mathbb{R}^d$ be contracting maps.

Theorem (Hutchinson 1981)

There is a unique nonemtpy compact set $X \subseteq \mathbb{R}^d$ such that

$$X = f_1(X) \cup \cdots \cup f_n(X).$$

- **Definition of "fractal"**

Example 1

- $f_1 : x \mapsto \frac{1}{2}x$
- $f_2 : x \mapsto \frac{1}{2}x + \binom{1/2}{0}$
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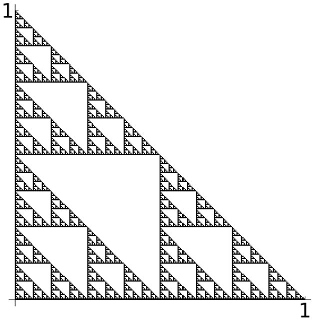

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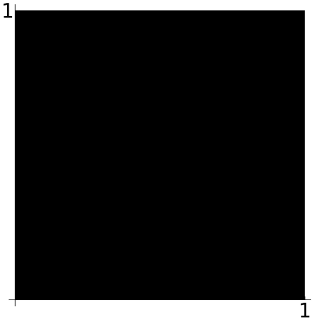

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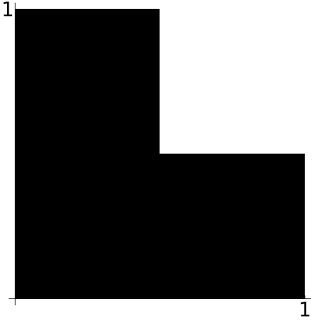

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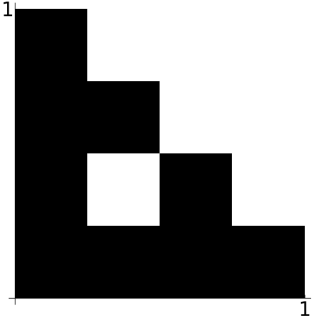

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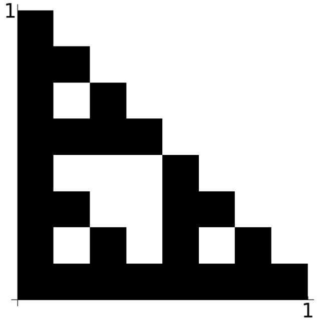

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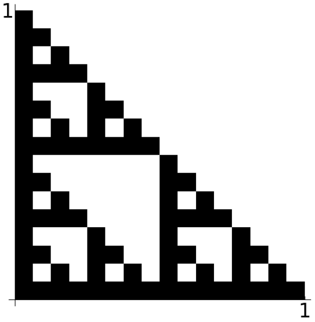

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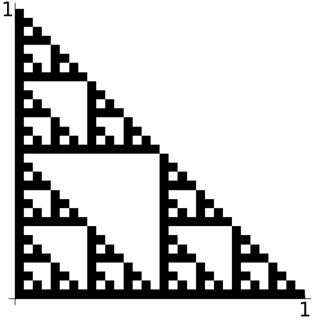

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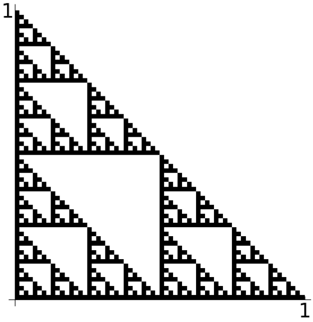

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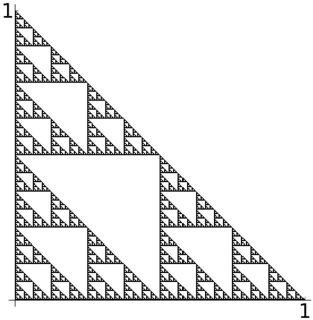

Self-affine sets

- Let $A \in \mathcal{M}_d(\mathbb{Z})$ be an expanding matrix (eigenvalues $|\lambda_i| > 1$)
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$$X = \bigcup_{d \in \mathcal{D}} A^{-1}(X + d).$$

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Very particular kind of IFS:

- Affine maps given by integers
- Common matrix for all maps

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- $A = (10)$
- $\mathcal{D} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

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Meaning of "digits":

$$X = \left\{ \sum_{k=1}^{\infty} A^{-k} d_k : (d_k)_{k \geqslant 1} \in \mathcal{D}^{\mathbb{N}} \right\}$$

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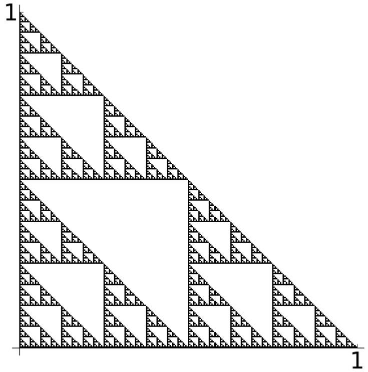
$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} X \;=\; X \;\cup\; \left(X + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \;\cup\; \left(X + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$



Sierpiński variant 1

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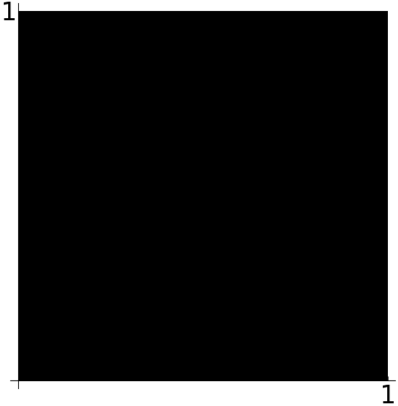

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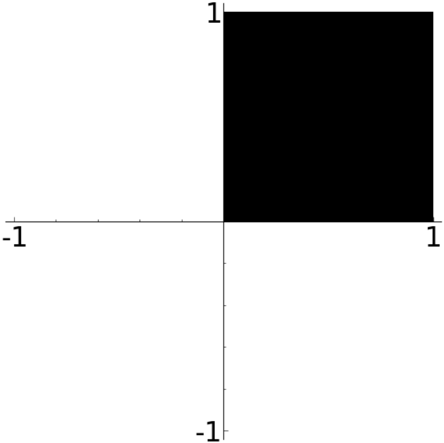

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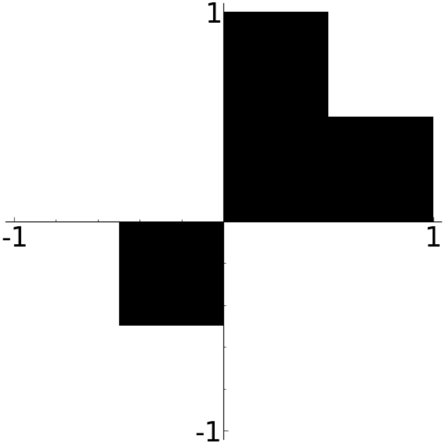

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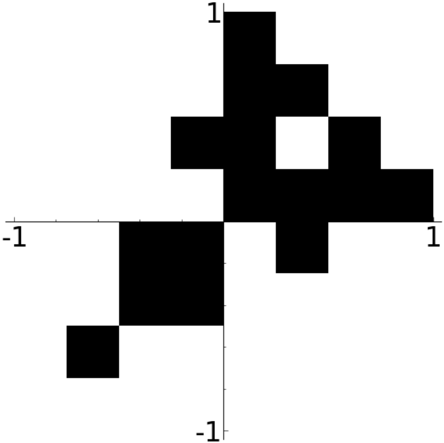

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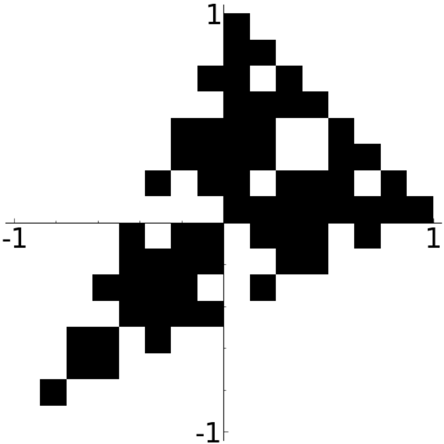

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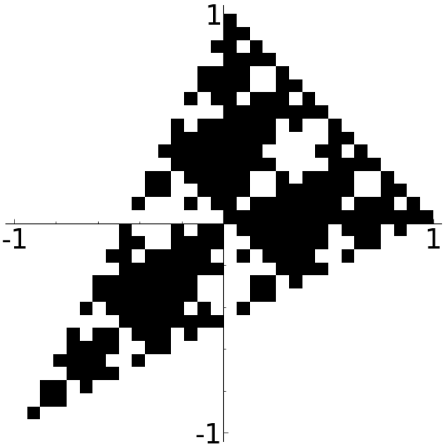

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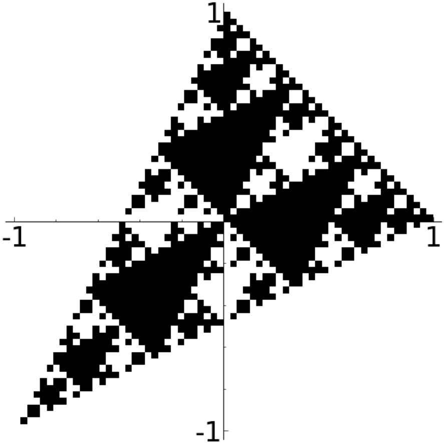

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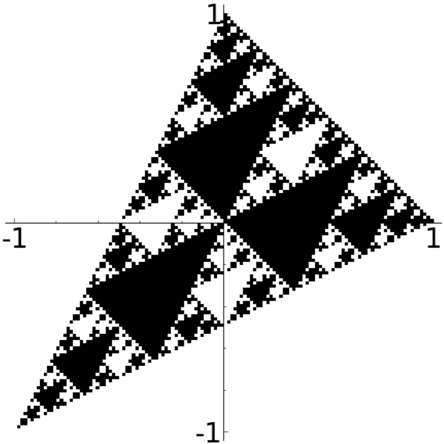

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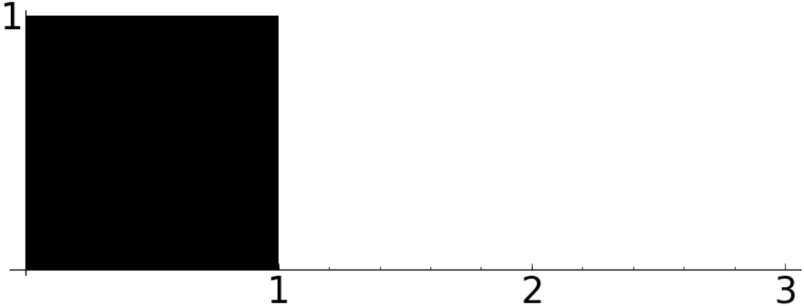

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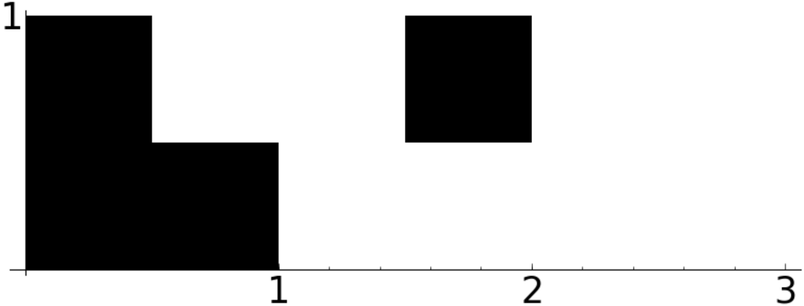

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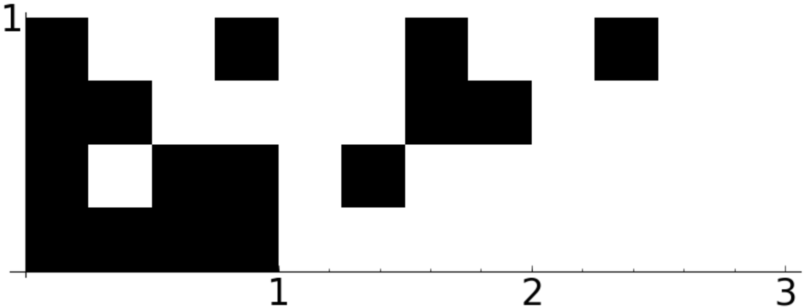

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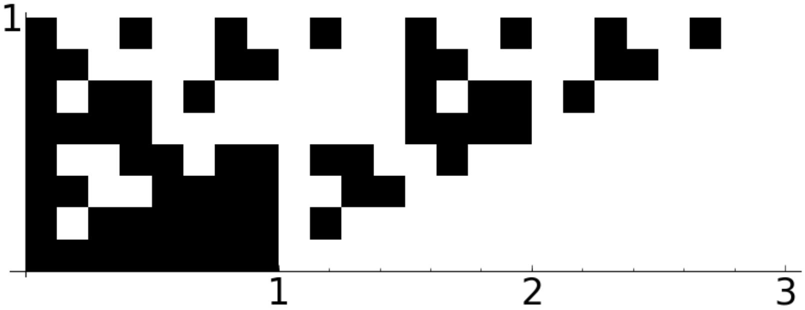

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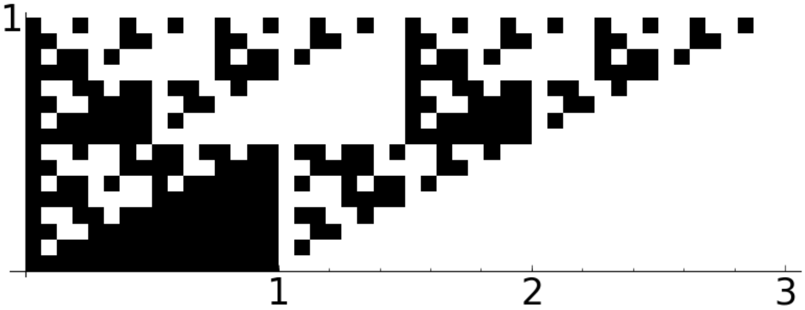

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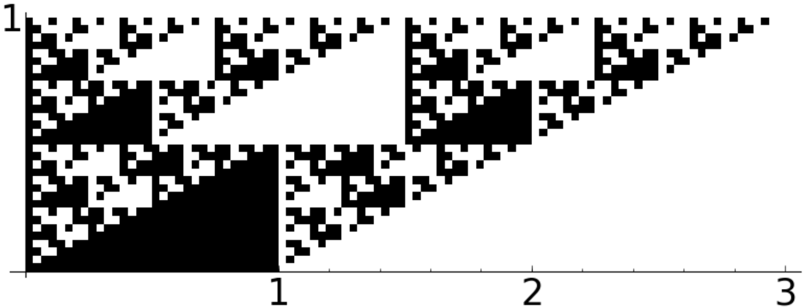

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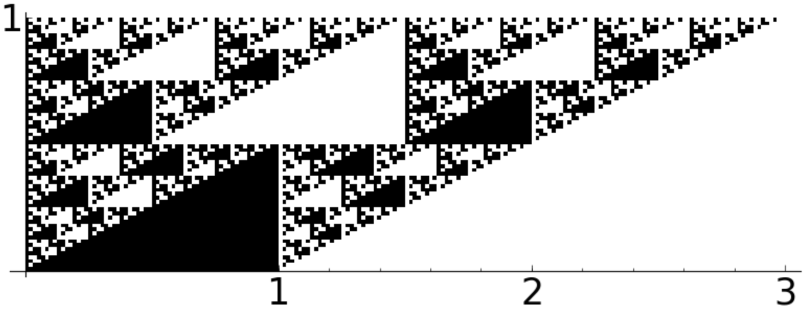

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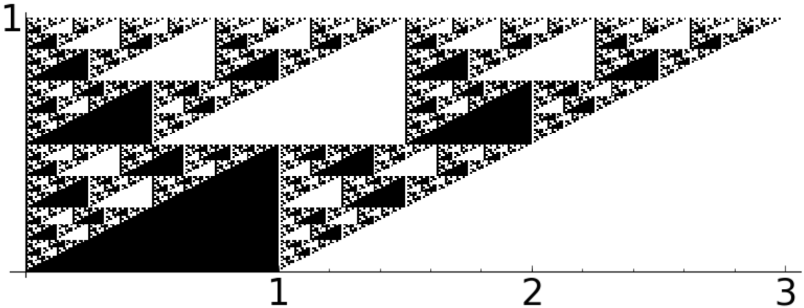

Self-affine tiles

Definition

$\mathcal{D}$ is a **standard** set of digits if:

- $|\mathcal{D}| = |\det(A)|$
- $\mathcal{D}$ is a complete set of representatives in $\mathbb{Z}^d/A\mathbb{Z}^d$.

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Theorem [Bandt, Lagarias-Wang]

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- Many good properties and algorithms

Nonstandard $\mathcal{D}$

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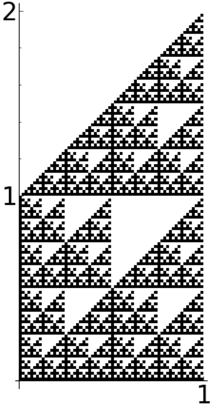

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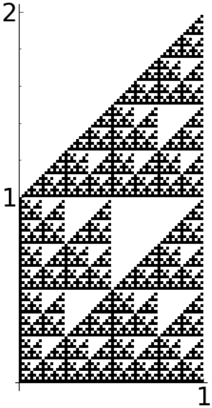

X **has empty interior (why?)**

Nonstandard $\mathcal{D}$

Indeed,

$$\left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right) X = X + \left\{ \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right) \right\}$$

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so

$$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} X = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} X + \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix} \right\}$$

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$$\begin{aligned} \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} X &= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} X + \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix} \right\} \\ &= X + \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\} \\ &\qquad + \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix} \right\} \end{aligned}$$

Nonstandard $\mathcal{D}$

Indeed,

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} X = X + \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$$

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so $16\mu(X) \leqslant 15\mu(X)$

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so $16\mu(X) \leqslant 15\mu(X)$

so $\mu(X) = 0$

Nonstandard $\mathcal{D}$

$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$
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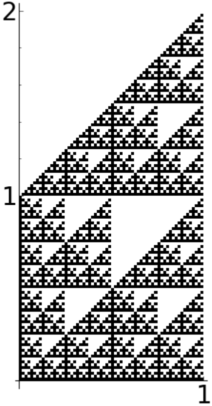

X **has empty interior**

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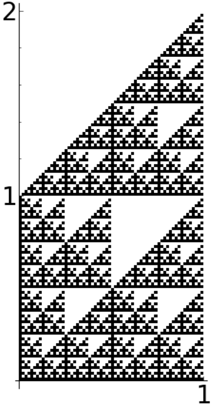

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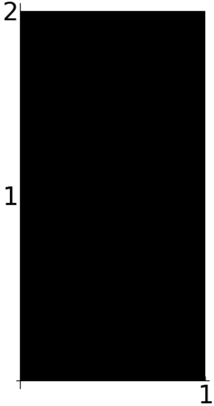

X **has nonempty interior**

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We like tiles and their good properties :

What conditions must we put on $\mathcal{D}$
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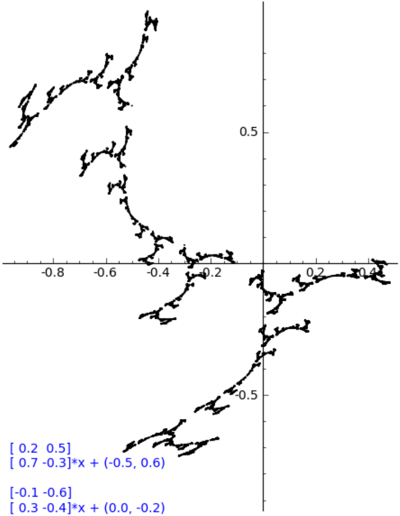
- Several other works in this direction...
- The real question: **Is it decidable?**

Beyond self-affine tiles: **affine IFS**

- Several contractions $A_1, \ldots, A_i$ instead of just A
- Arbitrary affine mappings $f_i(x) = A_i x + v_i$

Beyond self-affine tiles: affine IFS



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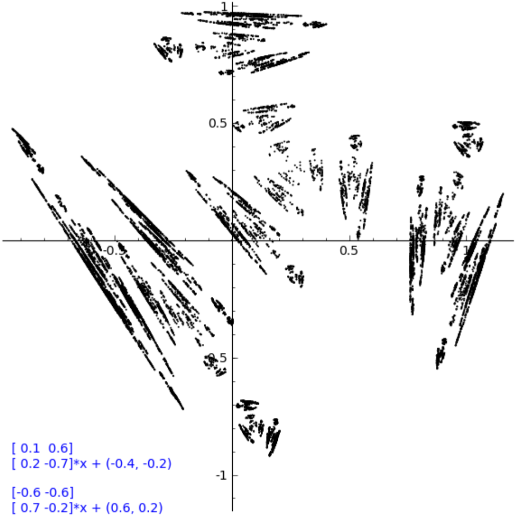

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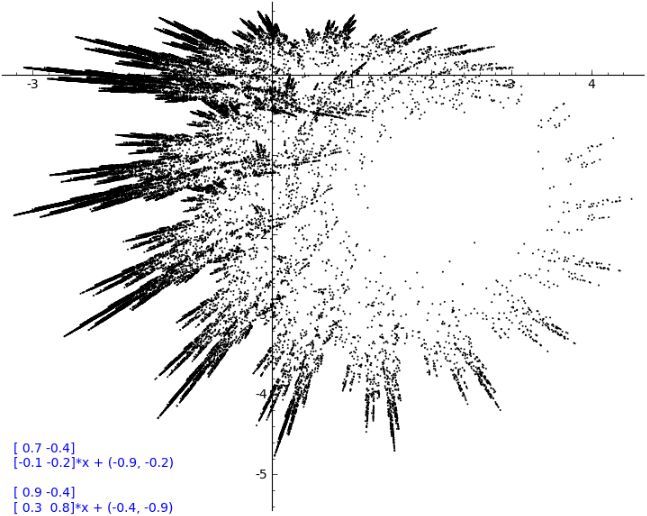

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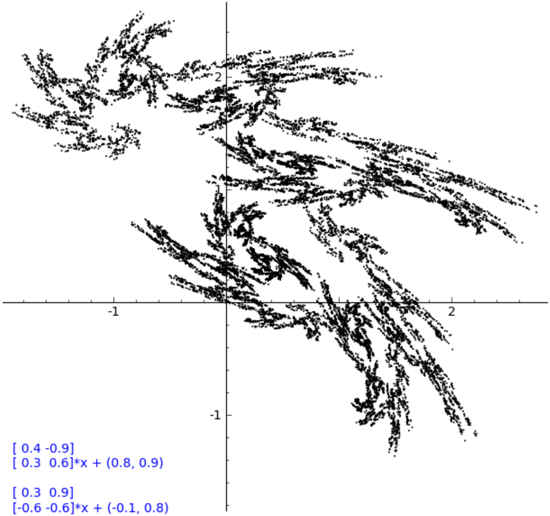

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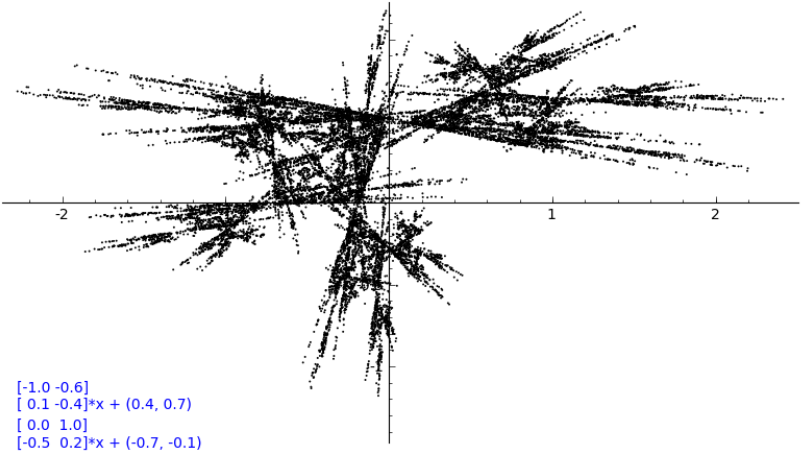

[-1.0 -0.6]
[0.1 -0.4]*x + (0.4, 0.7)

[0.0 1.0]
[-0.5 0.2]*x + (-0.7, -0.1)

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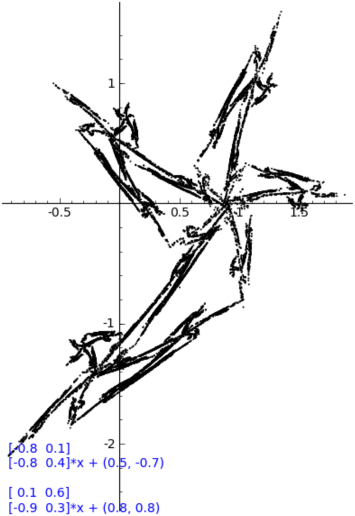

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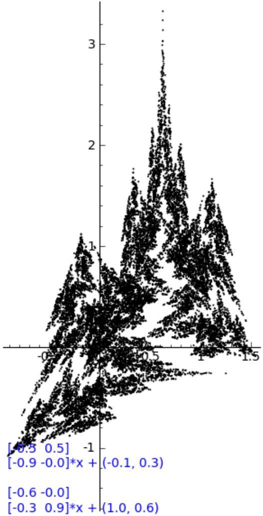

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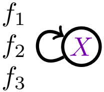
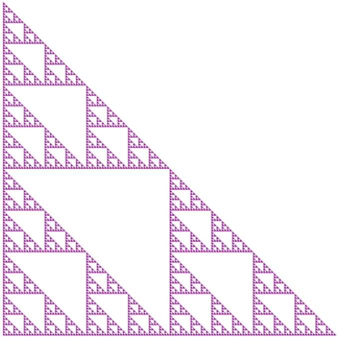
- Arbitrary affine mappings $f_i(x) = A_i x + v_i$
- Several sets $X_1, \ldots, X_n$ instead of just X (**Graph**-IFS)

Graph-IFS (GIFS)

$$f_1(x) = x/2$$
$$f_2(x) = x/2 + \binom{1/2}{0}$$
$$f_3(x) = x/2 + \binom{0}{1/2}$$

$$X = f_1(X) \cup f_2(X) \cup f_3(X)$$





Graph-IFS (GIFS)

$$f_1(x) = x/2 \qquad f_4(x) = x/2 + \binom{1/2}{1/2}$$
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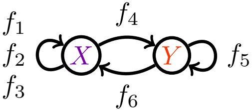

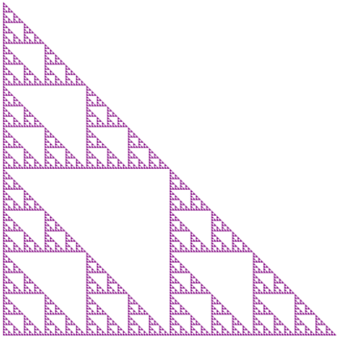

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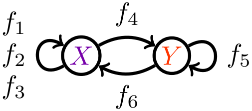

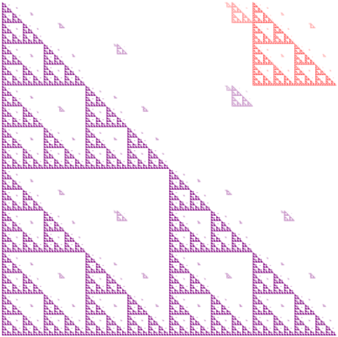

Undecidability result

Theorem [J-Kari 2013]

For 2D affine GIFS with 3 states (with coefficients in $\mathbb{Q}$):

- $= [0,1]^2$ is undecidable
- empty interior is undecidable

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- Computational tools: multi-tape automata

Multi-tape automata

d-tape automaton:

- alphabet $\mathcal{A} = A_1 \times \cdots \times A_d$
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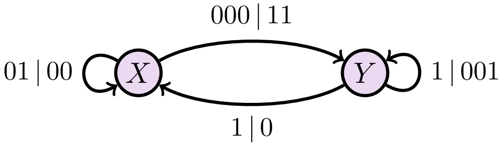
- transitions $\mathcal{Q} \times (A_1^+ \times \cdots \times A_d^+) \to \mathcal{Q}$



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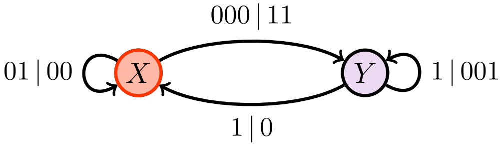

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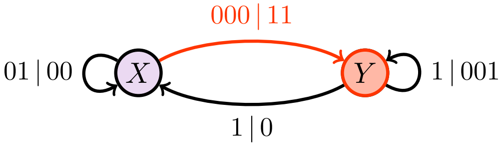

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000
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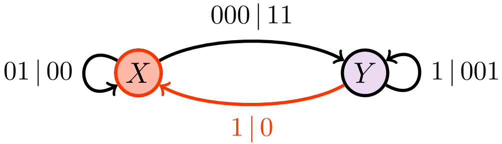

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110

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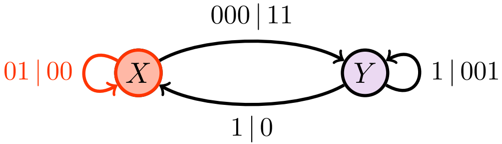

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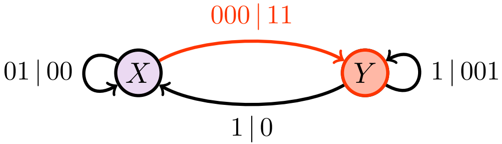

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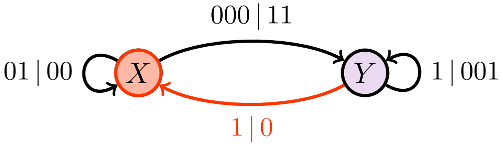


$\mathcal{A} = \{0,1\} \times \{0,1\}$
$\mathcal{Q} = \{X, Y\}$

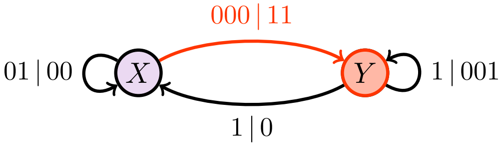
Accepted infinite word starting from X:

0001010001
11000110

Multi-tape automata

d-tape automaton:

- alphabet $\mathcal{A} = A_1 \times \cdots \times A_d$
- states $\mathcal{Q}$
- transitions $\mathcal{Q} \times (A_1^+ \times \cdots \times A_d^+) \to \mathcal{Q}$



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1100011011

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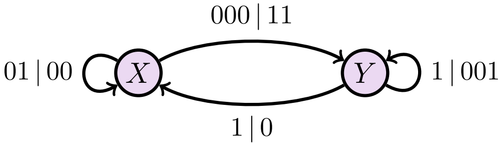

$\mathcal{A} = \{0,1\} \times \{0,1\}$
$\mathcal{Q} = \{X, Y\}$

Accepted infinite word starting from X:

$$\begin{matrix} 0001010001000\ldots \\ 1100011011\ldots \end{matrix} \in \mathcal{A}^{\mathbb{N}} = (\{0,1\} \times \{0,1\})^{\mathbb{N}}$$

Multi-tape automaton $\longmapsto$ GIFS

Transition 

(tape alphabets A_1, A_2)

Multi-tape automaton $\longmapsto$ GIFS

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(tape alphabets A_1, A_2)

$\longmapsto$ Mapping $f\binom{x}{y} = \begin{pmatrix} |A_1|^{-|u|} & 0 \\ 0 & |A_2|^{-|v|} \end{pmatrix}\binom{x}{y} + \begin{pmatrix} 0.u_1 \dots u_{|u|} \\ 0.v_1 \dots v_{|v|} \end{pmatrix}$

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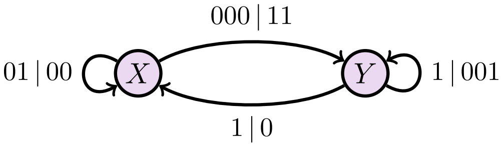
Automaton:



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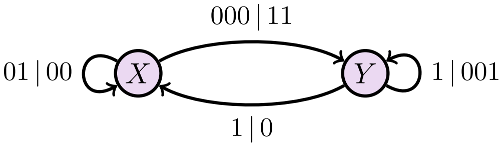
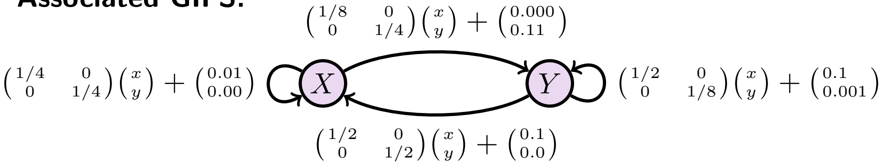


Associated GIFS:



Multi-tape automaton $\longmapsto$ GIFS

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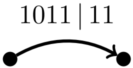
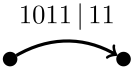
Multi-tape automaton $\longmapsto$ GIFS

- 

- $$f\binom{x}{y} = \begin{pmatrix} 1/16 & 0 \\ 0 & 1/4 \end{pmatrix} \begin{pmatrix} 0.x_1 x_2 \ldots \\ 0.y_1 y_2 \ldots \end{pmatrix} + \begin{pmatrix} 0.1011 \\ 0.11 \end{pmatrix}$$

Multi-tape automaton $\longmapsto$ GIFS

- 

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$$= \begin{pmatrix} 0.0000x_1x_2\ldots \\ 0.00y_1y_2\ldots \end{pmatrix} + \begin{pmatrix} 0.1011 \\ 0.11 \end{pmatrix}$$

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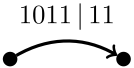
- 

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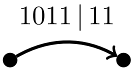
- 

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Key correspondence

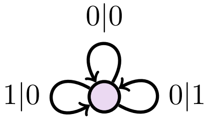
GIFS fractal associated with automaton $\mathcal{M}$

$$=$$

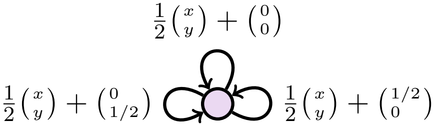
$$\left\{ \begin{pmatrix} 0.x_1x_2\dots \\ 0.y_1y_2\dots \end{pmatrix} \in \mathbb{R}^2 \ : \ \begin{pmatrix} x_1x_2\dots \\ y_1y_2\dots \end{pmatrix} \text{ accepted by } \mathcal{M} \right\}$$

Multi-tape automaton $\longmapsto$ GIFS

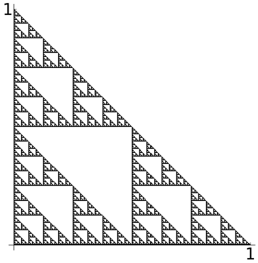
Example:



Automaton



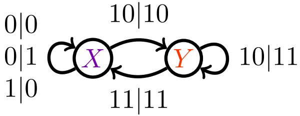
GIFS

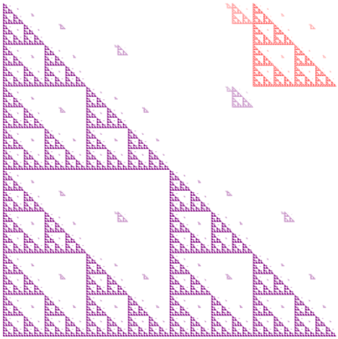


$$= \left\{ \begin{pmatrix} 0.x_1 x_2 \ldots \\ 0.y_1 y_2 \ldots \end{pmatrix} : (x_n, y_n) \neq (1,1),\ \forall n \geqslant 1 \right\}$$

Multi-tape automaton $\longmapsto$ GIFS

Example:





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Languages properties	**GIFS properties**
$\exists$ configurations with $=$ tapes	Intersects the diagonal [Dube]
Is universal	Is equal to $[0,1]^d$
Has universal prefixes	Has nonempty interior
?	Is connected
?	Is totally disconnected
Compute language entropy	Compute fractal dimension

Multi-tape automaton $\longleftrightarrow$ GIFS

Theorem [Dube 1993]

It is undecidable if X intersects the diagonal.

Proof idea:

- $X \cap \{(x,x) : x \in [0,1]\} \neq \varnothing$
  $\iff$ Automaton accepts a word of the form $\binom{0.x_1x_2\ldots}{0.x_1x_2\ldots}$
- Reduce the Post-correspondence problem

Language universality $\iff$ nonempty interior

Fact 1: $\mathcal{M}$ is universal $\iff$ $X = [0,1]^d$

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Fact 2: $\mathcal{M}$ is prefix-universal $\iff$ X has nonempty interior

- **Example (universal with prefix 1 but not universal):** one state, transitions 1, 10, 00 (one-dimensional)

$$
\begin{aligned}
f_1(x) &= x/2 + 1/2 \\
f_2(x) &= x/4 \\
f_3(x) &= x/4 + 1/2
\end{aligned}
$$

Undecidability results

Theorem [J-Kari 2013]

For 3-state, 2-tape automata:

- universality is undecidable
- prefix-universality is undecidable

Corollary

For 2D affine GIFS with 3 states (with coefficients in $\mathbb{Q}$):

- $= [0,1]^2$ is undecidable
- empty interior is undecidable

Conclusion & perspectives

Decidability of nonempty interior:

	IFS	2-state GIFS	$\geqslant 3$-state GIFS
dimension 1	**?**	**?**	**?**
dimension $\geqslant 2$	**?**	**?**	**Undecidable**

Conclusion & perspectives

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Merci pour votre attention

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