On the finiteness and the order problems for automaton (semi)groups

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Burnside

Milnor

\[ \text{Generating} \]

\[ \text{(semi)group} \]

\[ \text{Finiteness}: \text{undecidability} \]

\[ \text{Finiteness}: \text{criteria} \]

\[
\begin{array}{c|c}
  0 & 0 \\
  1 & 1 \\
\end{array}
\]

\[
\begin{array}{c|c}
  1 & 1 \\
  0 & 0 \\
\end{array}
\]

\[
\begin{array}{c|c}
  0 & 1, 1 | 0 \\
  1 & 1 | 0 \\
\end{array}
\]
Burnside

Milnor [growth]

Generating a (semi)group

Finiteness: undecidability

Finiteness: criteria

Burnside

\[ 0|1, 1|0 \]

\[ 0|0 \]

\[ 1|1 \]

\[ 0|0 \]

\[ 1|1 \]
Generating a (semi)group

Finiteness: undecidability

Finiteness: criteria

Burnside

Milnor [growth]

\[
\begin{align*}
0|1, 1|0 \\
0|0, 0|0 \\
1|1, 1|1 \\
0|0, 0|0 \\
1|1, 1|1 \\
0|0, 0|0 \\
\end{align*}
\]
Is a finitely generated group whose all elements have finite order necessarily finite?
Is a finitely generated group whose all elements have finite order necessarily finite?

1902

1964

Burnside

Golod

Shafarevich
Is a finitely generated group whose all elements have finite order necessarily finite?

No!

Even if the orders are bounded.
Is a finitely generated group whose all elements have finite order necessarily finite?

With automaton groups?

1902
1961

Glushkov
Burnside

Aleshin
Grigorchuk
Is a finitely generated group whose all elements have finite order necessarily finite?

With automaton groups?

Aleshin 1972

Glushkov Burnside 1961
Is a finitely generated group whose all elements have finite order necessarily finite?

With automaton groups?

1902

1961

1972

Aleshin

1980

Grigorchuk

Glushkov

Burnside
Milnor [growth]

Generating a (semi)group

Finiteness : undecidability

Finiteness : criteria

Burnside

\[\begin{align*}
0|1, 1|0 \\
0|0, 0|0 \\
1|1, 1|1
\end{align*}\]
\[ \mathbb{Z}^2 = \langle a = (0, 1), b = (1, 0) \mid ab = ba \rangle \]
Growth

\[ \mathbb{Z}^2 = \langle a = (0, 1), b = (1, 0) \mid ab = ba \rangle \]

\[ \gamma(0) = 1 \]
\( \mathbb{Z}^2 = \langle a = (0, 1), b = (1, 0) \mid ab = ba \rangle \)

\( \gamma(0) = 1 \)
\( \gamma(1) = 5 \)
\[ \mathbb{Z}^2 = \langle a = (0, 1), b = (1, 0) \mid ab = ba \rangle \]
\[ \mathbb{Z}^2 = \langle a = (0, 1), b = (1, 0) \mid ab = ba \rangle \]

\[ \gamma(0) = 1 \]
\[ \gamma(1) = 5 \]
\[ \gamma(2) = 13 \]
\[ \vdots \]
\[ \gamma(n) = 2n^2 + 2n + 1 \]
Growth

\[ F_2 = \langle a, b \rangle \]
$F_2 = \langle a, b \rangle$

$\gamma(0) = 1$
Growth

$$\mathbb{F}_2 = \langle a, b \rangle$$

$$\gamma(0) = 1$$
$$\gamma(1) = 5$$
\[ \mathbb{F}_2 = \langle a, b \rangle \]

\[ \gamma(0) = 1 \]
\[ \gamma(1) = 5 \]
\[ \gamma(2) = 17 \]
$\mathbb{F}_2 = \langle a, b \rangle$

$\gamma(0) = 1$
$\gamma(1) = 5$
$\gamma(2) = 17$
$\vdots$
$\gamma(n) = 2 \times 3^n - 1$

Growth
bounded growth: finite groups

polynomial growth: $\mathbb{Z}^d$, abelian groups

exponential growth: $\mathbb{F}_d$
bounded growth: finite groups

polynomial growth: $\mathbb{Z}^d$, abelian groups

1968

Is there an in between?

exponential growth: $\mathbb{F}_d$

Milnor
bounded growth: finite groups

polynomial growth: $\mathbb{Z}^d$, abelian groups

1968

Is there an in between?

exponential growth: $\mathbb{F}_d$

1983

Yes!

Grigorchuk

Milnor
(Semi)Groups generated by automata

A

0|1, 1|0
X

1|1
Z

0|0

0|0

0|0

1|1

1|1

A
(Semi)Groups generated by automata

\[ \mathcal{A} \]

\[ \rho_x : \ 01000 \leftrightarrow 11100 \]
(Semi)Groups generated by automata

\[ \langle A \rangle \] is a semigroup.

\[ \rho_x : 01000 \leftrightarrow 11100 \]

\[ \Sigma^* \rightarrow \Sigma^* \]
(Semi)Groups generated by automata

\[ A \]

\[ \langle A \rangle_+ \] semigroup

\[ \rho_x : \quad 01000 \leftrightarrow 11100 \]

\[ \Sigma^* \rightarrow \Sigma^* \]

- deterministic [fonctional]
- complete [defined all over \( \Sigma^* \)]
(Semi)Groups generated by automata

\[ A \]

\[ \langle A \rangle_+ \] semigroup
\[ \langle A \rangle \] group

\[ \rho_x : \quad 01000 \leftrightarrow 11100 \]
\[ \Sigma^* \rightarrow \Sigma^* \]

- deterministic [fonctional]
- complete [defined all over \( \Sigma^* \)]
- the states permute the alphabet [for groups]
(Semi)Groups generated by automata

\[ A \]

\[ \langle A \rangle_+ \text{ semigroup} \]
\[ \langle A \rangle \text{ group} \]

acts on finite words
\[ \rho_x : 01000 \mapsto 11100 \]
(Semi)Groups generated by automata

\[ \langle A \rangle \] semigroup
\[ \langle A \rangle \] group

- acts on finite words
  \[ \rho_x : 01000 \mapsto 11100 \]
- acts on infinite words
  \[ \rho_x : 01000^\omega \mapsto 11(100)^\omega \]
(Semi)Groups generated by automata

\[ \langle A \rangle \text{ acts on finite words} \]
\[ \rho_x : 01000 \mapsto 11100 \]

\[ \langle A \rangle \text{ acts on infinite words} \]
\[ \rho_x : 01000^\omega \mapsto 11(100)^\omega \]

\[ \langle A \rangle \text{ acts on the regular rooted tree} \]

\[ \rho_x : 01010101 \mapsto 01100110 \]
Some examples
Some examples

\[ Z / 2 \cong Z \times Z / 2 \]

\[ \rho_a : \quad 001110010101 \ldots \quad 101110010101 \ldots \]

\[ \rho_b : \quad \downarrow \quad \downarrow \]

\[ 110001101010 \ldots \quad 010001101010 \ldots \]
Some examples

\[ \rho_a: 001110010101 \ldots \rightarrow 101110010101 \ldots \]

\[ \rho_b: 110001101010 \ldots \rightarrow 010001101010 \ldots \]

\[ \rho_b^2 = \text{id}_{\Sigma^*} \]
Some examples

\[ \rho_a : 001110010101 \ldots \downarrow \quad 101110010101 \ldots \]
\[ \rho_b : 001110010101 \ldots \downarrow \quad 101110010101 \ldots \]
\[ \rho^2_b = \text{id}_{\Sigma^*} \]
Some examples

\[ \rho_a : 001110010101 \ldots \quad \updownarrow \quad 101110010101 \ldots \]
\[ 010001101010 \ldots \quad \updownarrow \quad 110001101010 \ldots \]
\[ \rho^2_a = \text{id}_{\Sigma^*} \]

\[ \rho_b : 001110010101 \ldots \quad \updownarrow \quad 101110010101 \ldots \]
\[ 110001101010 \ldots \quad \updownarrow \quad 010001101010 \ldots \]
\[ \rho^2_b = \text{id}_{\Sigma^*} \]
Some examples

\[
\begin{align*}
\rho_a : & \quad 001110010101 \ldots \quad 101110010101 \ldots \\
& \quad 010001101010 \ldots \quad 110001101010 \ldots \\
\rho_b : & \quad 001110010101 \ldots \quad 101110010101 \ldots \\
& \quad 110001101010 \ldots \quad 010001101010 \ldots \\
\rho_b \rho_a = \rho_a \rho_b : & \quad 001110010101 \ldots \quad 101110010101 \ldots \\
& \quad 101110010101 \ldots \quad 001110010101 \ldots \\
\rho_a^2 = \text{id}_{\Sigma^*} \\
\rho_b^2 = \text{id}_{\Sigma^*}
\end{align*}
\]
Some examples

\[\begin{array}{c}
\rho_a : \\
\downarrow \\
010001101010 \ldots \\
\downarrow \\
110001101010 \ldots \\
\end{array}\]

\[\begin{array}{c}
\rho_b : \\
\downarrow \\
110001101010 \ldots \\
\downarrow \\
010001101010 \ldots \\
\end{array}\]

\[\rho_a = \text{id}_{\Sigma^*}\]

\[\rho_b = \text{id}_{\Sigma^*}\]

\[\rho_b \rho_a = \rho_a \rho_b : \\
\downarrow \\
101110010101 \ldots \\
\downarrow \\
001110010101 \ldots \\
\end{array}\]

\[(\rho_a \rho_b)^2 = \text{id}_{\Sigma^*}\]
Some examples

\[ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \]

\[ \rho_a : \]
\[
\begin{array}{c}
0011100101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101010101
Some examples

\[ N, Z : 111001010100 \rightarrow 111001010100 + 0 \]

\[ o : 111001010100 \rightarrow 000101010100 + 1 \]
Some examples

\[
\begin{align*}
z : 111001010100 \leftrightarrow 111001010100 \\
o : 111001010100 \leftrightarrow 000101010100
\end{align*}
\]
Some examples

\[ z : 111001010100 \mapsto 111001010100 \]

\[ o : 111001010100 \mapsto 000101010100 + 1 \]
Some examples

\[ z : 111001010100 \mapsto 111001010100 + 0 \]

\[ o : 111001010100 \mapsto 000101010100 + 1 \]
Some examples

\[ z : 111001010100 \leftrightarrow 111001010100 + 0 \]

\[ o : 111001010100 \leftrightarrow 000101010100 + 1 \]
Some examples

\[ z : 111001010100 \leftrightarrow 111001010100 + 0 \]

\[ o : 111001010100 \leftrightarrow 000101010100 + 1 \]
How automata can contribute

The word problem
How automata can contribute

The word problem

\[
\begin{align*}
\rho_{x | \Sigma^*} &= \rho_{y | \Sigma^*} \\
\end{align*}
\]
How automata can contribute

The word problem

\[
\begin{align*}
\{1, 2, 3\} 
\end{align*}
\]
How automata can contribute

The word problem
How automata can contribute

The word problem
How automata can contribute

The word problem

+ classical minimisation:

\[ [x] = [y] \iff \rho_{x|\Sigma^*} = \rho_{y|\Sigma^*} \]
How automata can contribute

The word problem

\[ \rho x_1 \cdots x_n \overset{?}{=} \rho y_1 \cdots y_{n+k} \]
How automata can contribute

The word problem

$$\rho x_1 \cdots x_n \overset{?}{=} \rho y_1 \cdots y_{n+k}$$
How automata can contribute

The word problem

\[ \rho x_1 \cdots x_n \overset{?}{=} \rho y_1 \cdots y_{n+k} \]
Burnside

Milnor [growth]

Generating a (semi)group

Finiteness: undecidability

Finiteness: criteria

Burnside

\( x \cdot y \cdot z \mid 0 \)

\( 1 \mid 1, 1 \mid 0 \)

\( 0 \mid 0, 1 \mid 1 \)

\( 1 \mid 1 \)
The finiteness of an automaton semigroup is undecidable. A lot of partial results
- families of automata where the finiteness is decidable
- sufficient or necessary conditions of finiteness
The finiteness of an automaton semigroup is undecidable.

A lot of partial results
- families of automata where the finiteness is decidable
- sufficient or necessary conditions of finiteness
Finiteness of automaton (semi)groups

Theorem [Gillibert’13]
The finiteness of an automaton semigroup is undecidable.

A lot of partial results
- families of automata where the finiteness is decidable
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Finiteness of automaton (semi)groups

Theorem [Gillibert’13]

The finiteness of an automaton semigroup is undecidable.

A lot of partial results

- families of automata where the finiteness is decidable
- sufficient or necessary conditions of finiteness

\[ 1 494 \, 186 \, 269 \, 970 \, 473 \, 680 \, 896 = 2^{64} \cdot 3^4 \approx 1.5 \times 10^{21} \]
Finiteness of automaton (semi)groups

Theorem [Gillibert’13]
The finiteness of an automaton semigroup is undecidable.

\[ 1494186269970473680896 = 2^{64} \cdot 3^4 \approx 1.5 \times 10^{21} \]

A lot of partial results
- families of automata where the finiteness is decidable
- sufficient or necessary conditions of finiteness
Finiteness of automaton (semi)groups

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A lot of partial results
- families of automata where the finiteness is decidable
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The finiteness of an automaton semigroup is undecidable.

A lot of partial results
- families of automata where the finiteness is decidable
- sufficient or necessary conditions of finiteness
General case: finiteness is undecidable [Gillibert'13]

Wang’s tile:

![Wang's tile](image-url)
General case: finiteness is undecidable [Gillibert’13]

Wang’s tiles:
General case: finiteness is undecidable [Gillibert’13]

Wang’s tiles:

It is undecidable if a NW-deterministic tiling has a valid tiling for the plane.
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General case: finiteness is undecidable [Gillibert’13]

Wang’s tiles: 

tiling
General case: finiteness is undecidable [Gillibert’13]

Wang’s tiles:

It is undecidable if a NW-deterministic tiling has a valid tiling for the plane.
General case: finiteness is undecidable [Gillibert’13]

Wang’s tiles:

NW-deterministic tiling
General case: finiteness is undecidable [Gillibert’13]

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Wang’s tiles:

NW-deterministic tiling
General case: finiteness is undecidable [Gillibert’13]

Wang’s tiles:

NW-deterministic tiling

[Kari’92] It is undecidable if a NW-deterministic tiling has a valid tiling for the plane.
General case: finiteness is undecidable [Gillibert’13]

Wang’s tiles: $\mathbb{W}$

$\mathbb{W}$ has a valid Wang tiling for the plane $\iff \langle A \rangle^+ = \infty$
General case: finiteness is undecidable [Gillibert’13]

Wang’s tiles: $\text{\begin{array}{cccccc} & & \square & & \square & & \square \\ & & \square & & \square & & \square \\ & & \square & & \square & & \square \end{array}} = T$

$\mathcal{A}_T : \text{\begin{array}{cccccc} & & \square & & \square & & \square \\ & & \square & & \square & & \square \\ & & \square & & \square & & \square \end{array}} \rightarrow \text{\begin{array}{cccccc} & & \square & & \square & & \square \\ & & \square & & \square & & \square \\ & & \square & & \square & & \square \end{array}}$
General case: finiteness is undecidable [Gillibert’13]

Wang’s tiles: \[ T \]

\[ \mathcal{A}_T : \]

\[ \Rightarrow \langle \mathcal{A}_T \rangle^+ \text{ is infinite} \]
General case: finiteness is undecidable [Gillibert’13]

Wang’s tiles: \( T \)

\[ A_T : \]

\[ \Rightarrow \langle A_T \rangle + \text{is infinite} \]
General case: finiteness is undecidable [Gillibert’13]

Wang’s tiles: \[
\begin{array}{cccc}
\text{\begin{tikzpicture}[scale=0.5]
\fill[red] (0,0) rectangle (1,1);
\fill[blue] (1,0) rectangle (2,1);
\fill[green] (0,1) rectangle (1,2);
\fill[red] (1,1) rectangle (2,2);
\end{tikzpicture}} & \text{\begin{tikzpicture}[scale=0.5]
\fill[blue] (0,0) rectangle (1,1);
\fill[green] (1,0) rectangle (2,1);
\fill[red] (0,1) rectangle (1,2);
\fill[green] (1,1) rectangle (2,2);
\end{tikzpicture}} & \text{\begin{tikzpicture}[scale=0.5]
\fill[green] (0,0) rectangle (1,1);
\fill[red] (1,0) rectangle (2,1);
\fill[blue] (0,1) rectangle (1,2);
\fill[green] (1,1) rectangle (2,2);
\end{tikzpicture}} & \text{\begin{tikzpicture}[scale=0.5]
\fill[red] (0,0) rectangle (1,1);
\fill[blue] (1,0) rectangle (2,1);
\fill[green] (0,1) rectangle (1,2);
\fill[blue] (1,1) rectangle (2,2);
\end{tikzpicture}} & \text{\begin{tikzpicture}[scale=0.5]
\fill[green] (0,0) rectangle (1,1);
\fill[red] (1,0) rectangle (2,1);
\fill[blue] (0,1) rectangle (1,2);
\fill[red] (1,1) rectangle (2,2);
\end{tikzpicture}}
\end{array}
\]

\( = T \)

\[A_T:\]

\[\langle A_T \rangle_+ \text{ is infinite}\]

\[T \text{ has a valid Wang tiling for the plane } \iff \langle A_T \rangle_+ \text{ is infinite}\]
$T$ has a valid tiling for the plane

$T$ has a valid Wang tiling for the plane $\iff \langle \mathcal{A}_T \rangle_+ \text{ is infinite}$

[Gillibert’13]
$T$ has a valid tiling for the plane $\iff \langle \mathcal{A}_T \rangle_+ \text{ is infinite}$

[Gillibert’13]
$T$ has a valid tiling for the plane $\iff \langle \mathcal{A}_T \rangle_+ \text{ is infinite}$

[Gillibert’13]
$T$ has a valid tiling for the plane $\iff \langle A_T \rangle_+$ is infinite
$T$ has a valid tiling for the plane \iff $\langle \mathcal{A}_T \rangle_+$ is infinite

[Gillibert’13]

$T$ has a valid Wang tiling for the plane $\iff \langle \mathcal{A}_T \rangle_+$ is infinite
$T$ has a valid tiling for the plane

$\iff \langle \mathcal{A}_T \rangle_+ \text{ is infinite}$

[Gillibert'13]
$T$ has a valid tiling for the plane

$T$ has a valid Wang tiling for the plane $\iff \langle \mathcal{A}_T \rangle_+ \text{ is infinite}$
$T$ has a valid tiling for the plane ⇐⇒ $\langle A_T \rangle_+ \text{ is infinite}$

[Gillibert’13]
$T$ has a valid tiling for the plane $\iff \langle \mathcal{A}_T \rangle_+ \text{ is infinite}$

[Gillibert’13]
$T$ has a valid tiling for the plane ⇔ $\langle A_T \rangle_+$ is infinite

[Gillibert’13]

$T$ has a valid Wang tiling for the plane ⇔ $\langle A_T \rangle_+$ is infinite
General case: finiteness is undecidable [Gillibert’13]

Wang’s tiles: \[\begin{array}{cccc}
\text{\textcolor{red}{\textbullet} \textcolor{green}{\textbullet} \textcolor{blue}{\textbullet} \textcolor{red}{\textbullet}}
\end{array}\] = \(T\)

\(A_T:\)

\(\begin{array}{cccc}
\text{\textcolor{red}{\textbullet} \textcolor{blue}{\textbullet} \textcolor{green}{\textbullet} \textcolor{red}{\textbullet}}
\end{array}\)

\(\begin{array}{cccc}
\text{\textcolor{red}{\textbullet} \textcolor{blue}{\textbullet} \textcolor{green}{\textbullet} \textcolor{red}{\textbullet}}
\end{array}\)

\(\begin{array}{cccc}
\text{\textcolor{red}{\textbullet} \textcolor{blue}{\textbullet} \textcolor{green}{\textbullet} \textcolor{red}{\textbullet}}
\end{array}\)

\(\begin{array}{cccc}
\text{\textcolor{red}{\textbullet} \textcolor{blue}{\textbullet} \textcolor{green}{\textbullet} \textcolor{red}{\textbullet}}
\end{array}\)

\(\begin{array}{cccc}
\text{\textcolor{red}{\textbullet} \textcolor{blue}{\textbullet} \textcolor{green}{\textbullet} \textcolor{red}{\textbullet}}
\end{array}\)
Generating a (semi)group

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Milnor [growth]

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Finiteness of automaton (semi)groups

Theorem [Gillibert’13]
The finiteness of an automaton semigroup is undecidable.

A lot of partial results
- families of automata where the finiteness is decidable
- sufficient or necessary conditions of finiteness
The dual automaton

inverting the letters and the states roles
The dual automaton

inverting the letters and the states roles

\[ \mathcal{A} \]

\[ d(\mathcal{A}) \]

\[ \begin{align*}
0 & \rightarrow 1, 1 \rightarrow 0 \\
1 & \rightarrow 1, 1 \rightarrow 0
\end{align*} \]
The dual automaton

inverting the letters and the states roles

\( \mathcal{A} \)

\( \delta(\mathcal{A}) \)

\[ \langle \delta(\mathcal{A}) \rangle_+ \text{ finite } \iff \langle \mathcal{A} \rangle_+ \text{ finite} \]

\( \langle \delta(\mathcal{A}) \rangle_+ : \) words acting on the states of \( \mathcal{A}^n \)
The dual automaton

\[ \delta_i : \text{productions of } \partial(A) \]

\[ \langle \partial(A) \rangle_+ \text{ finite } \rightarrow \langle A \rangle_+ \text{ finite} \]

\[ \langle \partial(A) \rangle_+ : \text{words acting on the states of } A^n \]
The dual automaton

\( \delta_i : \text{productions of } \varnothing(A) \)

\( A : \)

\( i|\rho_x(i) \quad \delta_i(x) \)

\( x \rightarrow \delta_i(x) \)

\( \rho_x(i) \)

\( y \rightarrow 1|1 \)

\( \langle \varnothing(A) \rangle_+ \text{ finite } \rightarrow \langle A \rangle_+ \text{ finite} \)

\( \langle \varnothing(A) \rangle_+ : \text{words acting on the states of } \mathcal{A}^n \)
The dual automaton

\[ \delta_i : \text{productions of } \partial(A) \]

\[
\begin{array}{c}
  \text{w} \\
  \rho_w(u_1) \\
  \delta_{u_1}(w) \\
  \rho\delta_{u_1}(w)(u_2) \\
  \delta_{u_1}u_2\ldots u_{n-1}(w) \\
  \rho\delta_{u_1}\ldots u_{n-1}(w)(u_n) \\
\end{array}
\]

\[ \langle \partial(A) \rangle_+ \text{ finite } \Rightarrow \langle A \rangle_+ \text{ finite} \]

\[ \langle \partial(A) \rangle_+ : \text{words acting on the states of } A^n \]
The dual automaton

\[ \delta_i : \text{productions of } \varnothing(A) \]

\[
\begin{array}{c}
\overset{u_1}{w} \rightarrow \delta_{u_1}(w) \quad \overset{u_2}{\cdots} \rightarrow \delta_{u_1u_2\ldots u_{n-1}}(w) \\
\rho_w(u_1) \quad \rho_{\delta_{u_1}(w)}(u_2) \quad \rho_{\delta_{u_1u_2\ldots u_{n-1}}}(w)(u_n)
\end{array}
\]

Cayley(\langle \varnothing(A) \rangle^+) : \overset{i}{\delta_u} \rightarrow \delta_{ui}

\[ \langle \varnothing(A) \rangle^+ \text{ finite } \Rightarrow \langle A \rangle^+ \text{ finite} \]

\[ \langle \varnothing(A) \rangle^+ \text{ : words acting on the states of } A^n \]
The dual automaton

\[ \delta_i : \text{productions of } \partial(A) \]

\[ w \xrightarrow{\rho_w(u_1)} \delta_{u_1}(w) \xrightarrow{\rho_{\delta_{u_1}}(u_2)} \cdots \xrightarrow{\rho_{\delta_{u_1}\cdots u_{n-1}}(w)} u_n \]

Cayley(\langle \partial(A) \rangle_+) :

\[ \delta_u \xrightarrow{i \rho_{\delta_u}(i)} \delta_{ui} \]

\[ \langle \partial(A) \rangle_+ \text{ finite } \Rightarrow \langle A \rangle_+ \text{ finite} \]

\[ \langle \partial(A) \rangle_+ : \text{words acting on the states of } A^n \]
The dual automaton

\[ \langle \mathcal{A} \rangle_+ \text{ finite} \implies \langle \mathcal{A} \rangle_+ \text{ finite} \]

\[ \langle d(\mathcal{A}) \rangle_+ : \text{words acting on the states of } \mathcal{A}^n \]
The dual automaton

\[ \delta_i : \text{productions of } \partial(A) \]

\[ w \xrightarrow{\rho_w(u_1)} \delta_{u_1}(w) \xrightarrow{\rho_{\delta_{u_1}(w)}(u_2)} \cdots \xrightarrow{\rho_{\delta_{u_1 \cdots u_{n-1}(w)}(u_n)}} \delta_{u_1 \cdots u_{n-1}(w)} \]

Cayley(\langle \partial(A) \rangle_+) :

\[ \text{id} \xrightarrow{u_1 \rho_w(u_1)} \delta_{u_1} \xrightarrow{u_2 \rho_{\delta_{u_1}(w)}(u_2)} \delta_{u_1 u_2} \cdots \xrightarrow{u_n \rho_{\delta_{u_1 \cdots u_{n-1}(w)}(u_n)}} \delta_{u_1 \cdots u_{n-1}} \]

\[ \langle \partial(A) \rangle_+ \text{ finite } \rightarrow \langle A \rangle_+ \text{ finite} \]

\[ \langle \partial(A) \rangle_+ : \text{words acting on the states of } A^n \]
Theorem [Akhavi K Lombardy Mairesse Picantin’12]

$\mathcal{A}$ md-trivial $\implies \langle \mathcal{A} \rangle_+$ finite
Theorem [Akhavi K Lombardy Mairesse Picantin’12]

$\mathcal{A}$ m$d$-trivial $\implies \langle \mathcal{A} \rangle_+ \text{ finite}$
md-triviality

Theorem [Akhavi K Lombardy Mairesse Picantin’12]

\[ \mathcal{A} \text{ md-trivial} \implies \langle \mathcal{A} \rangle_+ \text{ finite} \]
Theorem [Akhavi K Lombardy Mairesse Picantin’12]

\[ A \text{-} \text{md-trivial} \implies \langle A \rangle_+ \text{ finite} \]
Theorem [Akhavi K Lombardy Mairesse Picantin’12]

$\mathcal{A}$ md-trivial $\implies \langle \mathcal{A} \rangle_+ \text{ finite}$
Theorem [Akhavi K Lombardy Mairesse Picantin’12]

\[ \mathcal{A} \text{ m\textdegree-trivial} \implies \langle \mathcal{A} \rangle_+ \text{ finite} \]
md-triviality

[Akhavi K Lombardy Mairesse Picantin’12]

$\mathcal{A}$ md-trivial $\iff \langle \mathcal{A} \rangle_+ \text{ finite}$
Theorem [K, STACS’13]

\( A \) 2-state inv.-rev. : \( A \) md-trivial \iff \( \langle A \rangle \) finite
Generating a (semi)group

Burnside

Milnor [growth]

Finiteness : undecidability

Finiteness : criteria

\[ x \]

\[ y \]

\[ z \]

\[ 0 \mid 0, 1 \mid 0 \]

\[ 0 \mid 0, 1 \mid 1 \]

\[ 1 \mid 1, 1 \mid 0 \]
The Burnside problem

Is a finitely generated group whose all elements have finite order necessarily finite?
Is a finitely generated group whose all elements have finite order necessarily finite?
The Burnside problem
A common point between these examples?

Is a finitely generated group whose all elements have finite order necessarily finite?

The diagram shows two graphs labeled with states and transitions. The states are represented by circles, and the transitions are shown as arrows between the states. The states are labeled with symbols and transitions labeled with 0 or 1, indicating different paths or operations between states. The diagram illustrates the concept of reversibility, as indicated by the notation “not reversible” and the arrangement of states and transitions.
The Burnside problem and the reversibility

A common point between these examples?

Is a finitely generated group whose all elements have finite order necessarily finite?

not reversible!

(letters are not permutations of the states)
The Burnside problem and the reversibility

**Natural question**

Can a reversible automaton generate an infinite Burnside group?
The Burnside problem and the reversibility

**Natural question**

Can a reversible automaton generate an infinite Burnside group?

impossible:

- 2-state [K., STACS’13]
- connected 3-state [K. Picantin Savchuk’15]
- no bireversible component [Godin K. Picantin, LATA’15]
\[ \rho_{x_i} : \Sigma^* \to \Sigma^* \]

\[ \rho_{x_1}x_2...x_n = \rho_{x_n} \circ \cdots \circ \rho_{x_1} \]

\[ \mathcal{A} \]
\( \rho_{x_i} : \Sigma^* \rightarrow \Sigma^* \)

\( \rho_{x_1 x_2 \ldots x_n} = \rho_{x_n} \circ \cdots \circ \rho_{x_1} \)
\[
\rho_{x_i} : \Sigma^* \rightarrow \Sigma^*
\]
\[
\rho_{x_1 x_2 \ldots x_n} = \rho_{x_n} \circ \cdots \circ \rho_{x_1}
\]
\[ \rho_{x_i} : \Sigma^* \rightarrow \Sigma^* \]
\[ \rho_{x_1x_2\ldots x_n} = \rho_{x_n} \circ \ldots \circ \rho_{x_1} \]
$\rho_{x_i} : \Sigma^* \rightarrow \Sigma^*$

$\rho_{x_1x_2...x_n} = \rho_{x_n} \circ \cdots \circ \rho_{x_1}$
\[ \rho_{x_i} : \Sigma^* \rightarrow \Sigma^* \]
\[ \rho_{x_1 x_2 \ldots x_n} = \rho_{x_n} \circ \cdots \circ \rho_{x_1} \]

\[ A \]
The connected components of the powers

\( \mathcal{A} \) reversible (letters induce permutations of the stateset)

**Finiteness problem**

\[ \langle \mathcal{A} \rangle \text{ is finite } \iff \text{the cc of the } \mathcal{A}^n \text{ are bounded} \]

**Order problem**

\[ \rho_u \text{ has finite order } \iff \text{the cc of the } u^n \text{ are bounded} \]
The connected components of the powers

$\mathcal{A}$ reversible (letters induce permutations of the stateset)

**Finiteness problem**

$\langle \mathcal{A} \rangle$ is finite $\iff$ the cc of the $\mathcal{A}^n$ are bounded

expressed as path properties in the orbit tree

**Order problem**

$\rho_u$ has finite order $\iff$ the cc of the $u^n$ are bounded
The labeled orbit tree
The labeled orbit tree

\[ \langle A \rangle \text{ is finite} \]

\[ \forall u \text{ of finite order} \]

\[ \forall \omega \text{ of finite order} \]

\[ \forall \omega \text{ of finite order} \]
The labeled orbit tree

\( \rho_u \) of finite order

the path of \( u^\omega \) ends with \( 1^\omega \)
First result: no bireversible component

[Godin K. Picantin, LATA’15]
First result: no bireversible component

[Godin K. Picantin, LATA’15]

- no path ends with $1^\omega$
- any element of $\langle A \rangle_+$ has infinite order
The labeled orbit tree

\[ \langle A \rangle \] is finite

all paths end with 1

\[ \rho \] of finite order

the path of \( u \) ends with 1
The labeled orbit tree

\[ \langle A \rangle \text{ is finite} \]

all paths end with 1\[ \omega \]

the path of \[ u \omega \] ends with 1\[ \omega \]
The labeled orbit tree

$\langle A \rangle$ is finite

All paths end with 1

The path of $u_\omega$ ends with 1

$xxxyx$
The labeled orbit tree

---

The labeled orbit tree is a mathematical structure used in group theory and combinatorics. In the diagram, the tree is labeled with elements from a group, and the paths indicate the group's operation. The tree structure helps visualize the relationships and properties of the group elements, such as associativity and closure. The labeled nodes and edges provide a clear representation of the group's behavior under the specified operation.
The labeled orbit tree

$yx \rightarrow$

$xxyx$
The labeled orbit tree

\[
yx \rightarrow
\]

\[
xxyx
\]
The labeled orbit tree

\[ yx \rightarrow \]

\[ xx \text{ } yx \]

[Diagram of a labeled orbit tree with nodes labeled 1, 2, and 3, and edges showing paths ending with 1.]

\[ \langle A \rangle \text{ is finite} \]

[All paths end with 1, \( \omega \rho u \text{ of finite order} \)]

[The path of \( u \omega \) ends with 1, \( \omega \).]
The labeled orbit tree

The labeled orbit tree
The labeled orbit tree

\[ \langle A \rangle \] is finite

\[ \forall \rho \ u \omega \text{ of finite order} \]

\[ \leftrightarrow \text{ the path of } u \omega \text{ ends with } 1 \]
Self-liftable paths

$k$-self-liftable paths
Self-liftable paths

A path is self-liftable if and only if it is the path of $u^\omega$ (for some $u$).
Second result: 3-state connected automata

[K. Picantin Savchuk’15]
Second result: 3-state connected automata

[K. Picantin Savchuk’15]

∃ 1-self-liftable paths $3^{\omega}$ or $3^n2^{\omega}$

∃ elements of infinite order
### Future work

#### Short term

- Can a reversible automaton generate an infinite Burnside group?
- Is the finiteness of an automaton group decidable?
- Is the order problem decidable in an automaton group?

#### Long term

- Is the growth type of an automaton group decidable?