Geometry and dynamics of reducible Pisot substitutions

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Reducible Pisot substitutions

Hokkaido substitution

\[ \sigma : 1 \mapsto 12, \ 2 \mapsto 3, \ 3 \mapsto 4, \ 4 \mapsto 5, \ 5 \mapsto 1 \]

\[ M_\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \]

\[ f(x)g(x) = (x^3 - x - 1)(x^2 - x + 1) \]

\( \beta > 1 \) dominant root of \( f(x) \) is Pisot \( \rightarrow |\beta'| < 1 \) for each conjugate \( \beta' \)

\( \sigma \) is a reducible unit Pisot substitution.
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\( \sigma \) is a \textbf{reducible unit Pisot} substitution.

\( M_\sigma \)-invariant decomposition:

\[ \mathbb{R}^5 = \mathbb{K}_\beta \oplus \mathbb{K}_n \]

\( M_\sigma \) is hyperbolic on \( \mathbb{K}_\beta \).
Reducible Pisot substitutions

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\[ M_{\sigma} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad f(x)g(x) = (x^3 - x - 1)(x^2 - x + 1) \]

\[ \beta > 1 \text{ dominant root of } f(x) \text{ is Pisot } \implies |\beta'| < 1 \text{ for each conjugate } \beta' \]

\[ \sigma \text{ is a reducible unit Pisot substitution.} \]

\( M_{\sigma} \)-invariant decomposition:

\[ \mathbb{R}^5 = \mathbb{K}_e \oplus \mathbb{K}_c \oplus \mathbb{K}_n \]

\( M_{\sigma} \) is expanding, contracting on \( \mathbb{K}_e \), resp. \( \mathbb{K}_c \).
Reducible Pisot substitutions

Hokkaido substitution

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\( M_\sigma \)-invariant decomposition:

\[ \mathbb{R}^5 \cong \mathbb{R} \oplus \mathbb{C} \oplus \mathbb{K}_n \]

\( M_\sigma \) is expanding, contracting on \( \mathbb{R} \), resp. \( \mathbb{C} \).
Substitutive system

For a primitive $\sigma$ with fixed point $u$ consider

$$X_\sigma = \{ S^k u : k \in \mathbb{Z} \}$$

where $S$ is the shift.

- $(X_\sigma, S)$ is a subshift.
- $(X_\sigma, S)$ is minimal, uniquely ergodic, with zero entropy.
- Pisot condition $\iff (X_\sigma, S)$ not weakly mixing.

Q: Does it have pure discrete spectrum?
\[ \sigma : 1 \mapsto 12, \ 2 \mapsto 3, \ 3 \mapsto 4, \ 4 \mapsto 5, \ 5 \mapsto 1 \]

\[ u = 1234511212312341234512345112 \cdots \]

- Projection of vertices of a broken line.

- Embedded beta numeration integers:

\[ \sum_{k \geq 0} \delta_c(d_k \beta^k), \ (d_k) \leq_{\text{lex}} (1) \beta \]

- GIFS directed by the prefix automaton of \( \sigma \).
Strong Coincidence Condition

For any \((i,j) \in \mathcal{A}^2\) there is \(k \in \mathbb{N}\) such that \(\sigma^k(i) \in p_1 a \mathcal{A}^*\) and \(\sigma^k(j) \in p_2 a \mathcal{A}^*\) for some \(a \in \mathcal{A}\) and words \(p_1, p_2 \in \mathcal{A}^*\) with \(l(p_1) = l(p_2)\).

If SCC holds then \((X_\sigma, S)\) is measurably conjugate to a domain exchange \((\mathcal{R}, E)\) defined a.e. by

\[
E : x \mapsto x + \pi_c(e_i), \quad \text{for } x \in \mathcal{R}(i)
\]
Having a periodic tiling by Rauzy fractals implies pure discrete spectrum. That’s true for all irreducible Pisot beta-substitutions (e.g. Tribonacci)!

\[ 1^\perp \text{ is the hyperplane orthogonal to (1, \ldots, 1) } \]

\[ \Lambda = \sum_{i \in A} \mathbb{Z}(e_i - e_1) \]
Having a periodic tiling by Rauzy fractals implies pure discrete spectrum. That’s true for all irreducible Pisot beta-substitutions (e.g. Tribonacci)!

\[ \sigma \xrightarrow{\sim} R \xrightarrow{\sim} 1^\perp / \Lambda \]

- \( 1^\perp \) is the hyperplane orthogonal to \((1, \ldots, 1)\)
- \( \Lambda = \sum_{i \in A} \mathbb{Z}(e_i - e_1) \)

Existence of necessary and sufficient geometrical, combinatorial and arithmetical conditions to get tilings.

**The Pisot Conjecture**
But... unfortunately the Hokkaido Rauzy fractal does not tile periodically!"
Reducible $\rightarrow$ linear dependencies

\[ \pi(e_1) = \pi(e_3) + \pi(e_4), \]
\[ \pi(e_5) = \pi(e_2) + \pi(e_3). \]

Combinatorially

\[ \chi : 1 \mapsto 34, 2 \mapsto 2, 3 \mapsto 3, 4 \mapsto 4, 5 \mapsto 32. \]
Reducible → linear dependencies

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Effect of \( \chi \) on a fixed point:

In this process we converted the substitution into an irreducible one!

Project now the vertices of the new broken line...
Reducible $\rightarrow$ linear dependencies

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Combinatorially

$$\chi : 1 \mapsto 34, 2 \mapsto 2, 3 \mapsto 3, 4 \mapsto 4, 5 \mapsto 32.$$  

Effect of $\chi$ on a fixed point:

$$\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 1 & 1 & 2 & 1 & 2 & 3 \\
3 & 4 & 2 & 3 & 4 & 3 & 2 & 3 & 4 & 3 & 4 & 2 & 3
\end{array}$$

In this process we converted the substitution into an irreducible one!

Project now the vertices of the new broken line...
• (\(\mathcal{R}, E\)) domain exchange on the original Hokkaido tile.

\[
E : \mathcal{R}(i) \mapsto \mathcal{R}(i) + \pi_c(e_i), \ i \in \mathcal{A}
\]

• (\(\widetilde{\mathcal{R}}, \widetilde{E}\)) is a toral translation, since it induces a periodic tiling of \(\mathbb{C} \mod \Lambda = \pi_c((e_4 - e_3)\mathbb{Z} + (e_4 - e_2)\mathbb{Z})\).

\[
\widetilde{E} : \widetilde{\mathcal{R}}(i) \mapsto \widetilde{\mathcal{R}}(i) + \pi_c(e_i), \ i \in \{2, 3, 4\}
\]

• \(E\) is the first return of \(\widetilde{E}\) on \(\mathcal{R}\).
Let $\Omega = \{S^k w : k \in \mathbb{N}\}$, where $w = \chi(u)$ is the coded fixed point of $\sigma$.

We have the following commutative diagram:

$$
\begin{array}{cccccc}
X_\sigma & \xrightarrow{\chi} & \Omega & \xrightarrow{\phi} & \tilde{R} & \xrightarrow{} & \mathbb{C}/\Lambda \\
S & \downarrow & S & \xrightarrow{E} & \tilde{E} & \downarrow & \\
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$\phi$ measure conjugation.
Let $\Omega = \{ S^k w : k \in \mathbb{N} \}$, where $w = \chi(u)$ is the coded fixed point of $\sigma$.

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$\Rightarrow (X_\sigma, S, \mu)$ is the first return of a toral translation.
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We can generalize what shown for the family of substitutions

$$
\sigma_t : 1 \mapsto 1^{t+1}2, \ 2 \mapsto 3, \ 3 \mapsto 4, \ 4 \mapsto 1^t5, \ 5 \mapsto 1
$$
(Arnoux, Ito 01) formalism for irreducible Pisot substitutions.

Action of the substitution on 1-dimensional faces $\rightarrow$ broken line
For $(x, a) \in \mathbb{Z}^d \times A$

$$E_1(\sigma)(x, a) = \sum_{\sigma(a) = pbs} (M_\sigma x + l(p), b)$$

Dual action on $(d - 1)$-dimensional faces:

$$E_1^*(\sigma)(x, a)^* = \sum_{\sigma(b) = pas} (M_\sigma^{-1}(x - l(p)), b)^*$$

Useful for:
1. generating Rauzy fractals as Hausdorff limits
2. producing stepped surfaces
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Useful for:

1. generating Rauzy fractals as Hausdorff limits.
2. producing stepped surfaces.
$$\mathcal{R}(a) = \lim_{k \to \infty} \pi_c(M^k_\sigma E_1^*(\sigma)^k(0, a)^*)$$
Hausdorff limits

\[ \mathcal{R}(a) = \lim_{k \to \infty} \pi_c(M^k_\sigma E^*_1(\sigma)^k(0, a)^*) \]
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Stepped surfaces

Set of coloured points “near” $K_c$:

$$\Gamma = \{(x, a) \in \mathbb{Z}^d \times A : x \in (K_c)^\geq, x - e_a \in (K_c)^<\}$$
Stepped surfaces

Set of coloured points “near” $K_c$:

$$\Gamma = \{(x, a) \in \mathbb{Z}^d \times A : x \in (K_c)\ge, x - e_a \in (K_c)\lt\}$$

- $E_1^*(\sigma)(\Gamma) = \Gamma \rightarrow$ self-replicating property (Kenyon).
- Aperiodic translation set (Delone set) for a self-replicating multiple tiling made of Rauzy fractals.
- Geometric representation as an arithmetic discrete model of the hyperplane $K_c$, whose projection is a polygonal tiling.
Reducible case: $n = \# \mathcal{A} > d = \deg(\beta)$.
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We want to work with \((d - 1)\)-dimensional faces! The dual map \( E^*_{n-d+1}(\sigma) \) will suit:

\[
E^*_{n-d+1}(\sigma)(x, a)^* = \sum_{b \xrightarrow{p} a} (M^{-1}_\sigma(x - l(p)), b)^*
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Reducible case: \( n = \# \mathcal{A} > d = \deg(\beta) \).

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Remarks:

- \( E^*_{n-d+1}(\sigma) \) acts on \( \binom{n}{n-d+1} \) oriented faces.
- If \( \sigma \) is irreducible \( n = d \) and \( E^*_{n-d+1}(\sigma) = E^*_1(\sigma) \).
- \( E_k(\sigma) \) and \( E^*_k(\sigma) \) commute in general with boundary and coboundary operators (Sano, Arnoux, Ito 2001).
- Similar approach for the study of a free group automorphism associated with a complex Pisot root (Arnoux, Furukado, Harriss, Ito 2011).
Under certain hypotheses we solve the problems of (Ei, Ito, Rao 06):

2. Hausdorff limit definition of renormalized patches of polygons.
3. Periodic (multiple) tiling.
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\[
\Gamma_\mathcal{U} := \bigcup_{k \geq 0} E_3^*(\sigma)^{5k}(\mathcal{U}), \quad \mathcal{U} = \{(0, 2 \wedge 3), (0, 2 \wedge 4), (0, 3 \wedge 4)\}.
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\[ \mathcal{R}(a) + \pi_c(x) = \lim_{k \to \infty} \pi_c(M^{k}_{\sigma} \mathcal{E}^{*}_{n-d+1}(\sigma)^{k}(x, a)^{*}) \]
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\[ R(\overline{a}) + \pi_c(x) = \lim_{k \to \infty} \pi_c(\mathcal{M}_\sigma^k \mathbf{E}_{n-d+1}(\sigma)^k(x, \overline{a})^*) \]
Under certain hypotheses we solve the problems of (Ei, Ito, Rao 06):

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\[
\mathcal{R}(\alpha) + \pi_c(x) = \lim_{k \to \infty} \pi_c(M^k_{\sigma} E^*_{n-d+1}(\sigma)^k(x, \alpha)^*)
\]
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\mathcal{R}(a) + \pi_c(x) = \lim_{k \to \infty} \pi_c(M_{\sigma}^k E_{n-d+1}(\sigma)^k(x, a)^*)
\]
Reducibility

Under certain hypotheses we solve the problems of (Ei, Ito, Rao 06):

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\[ R(a) + \pi_c(x) = \lim_{k \to \infty} \pi_c(M^k_{\sigma} E^\ast_{n-d+1}(\sigma)^k(x, a)^\ast) \]
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Strange examples

(Joint work with X. Bressaud and T. Jolivet)

The geometrical interpretation seems to get harder for these substitutions, not satisfying the strong coincidence condition:

\[ \sigma : 1 \mapsto 14, 2 \mapsto 32, 3 \mapsto 21, 4 \mapsto 3 \]

\[ \text{char}(M_\sigma) = (x - 1)(x^3 - x^2 - x - 1) \]
Strange examples

(Joint work with X. Bressaud and T. Jolivet)

The geometrical interpretation seems to get harder for these substitutions, not satisfying the strong coincidence condition:

$$\sigma : 1 \mapsto 213, \ 2 \mapsto 4, \ 3 \mapsto 5, \ 4 \mapsto 1, \ 5 \mapsto 21$$

$$\text{char}(M_\sigma) = (x^2 + x + 1)(x^3 - 2x^2 + x - 1)$$
Projection: \( \pi_{c,n} : \mathbb{R}^n \rightarrow K_c \oplus K_n \).
Lifting in the neutral space

Projection: $\pi_{c,n} : \mathbb{R}^n \rightarrow K_c \oplus K_n$. 

Criterion (depending on the prefix automaton of the substitution) to know whether we get only finitely many layers. In this case "NEW strong coincidence condition."
Lifting in the neutral space

Projection: \( \pi_{c,n} : \mathbb{R}^n \rightarrow K_c \oplus K_n \).

Criterion (depending on the prefix automaton of the substitution) to know whether we get only finitely many layers.

In this case \( \rightarrow \) NEW strong coincidence condition.
Gluing together

Projecting down suitably we can glue the subtiles together...
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Figure: Symbolic splitting associated with the irreducible substitution 
\( \tau : 1 \mapsto 12, 2 \mapsto 32, 3 \mapsto 1 \).

... and obtain the connection with an irreducible substitution.
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Figure: Symbolic splitting associated with the irreducible substitution $\tau : 1 \mapsto 12, 2 \mapsto 32, 3 \mapsto 1$.

...and obtain the connection with an irreducible substitution.

Philosophy: Dynamically the reducible substitutive system behaves exactly as the irreducible one, after identifying some letters / changing projection. Technique: symbolic splitting.
Figure: Rauzy fractal of the Hokkaido substitution in $K_c \oplus K_n$. The points distribute with logarithmic growth on a two-dimensional lattice.
Thank you!

Figure: Rauzy fractal of the Hokkaido substitution in $\mathbb{K}_c \oplus \mathbb{K}_n$. The points distribute with logarithmic growth on a two-dimensional lattice.