Geometry and dynamics of reducible Pisot substitutions

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LIAFA, Paris 7

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Hokkaido substitution

$$\sigma : 1 \mapsto 12,\ 2 \mapsto 3,\ 3 \mapsto 4,\ 4 \mapsto 5,\ 5 \mapsto 1$$

$$M_\sigma = \begin{pmatrix} 1&0&0&0&1 \\ 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \end{pmatrix}, \quad f(x)g(x) = (x^3 - x - 1)(x^2 - x + 1)$$

$\beta > 1$ dominant root of $f(x)$ is **Pisot** $\rightarrow$ $|\beta'| < 1$ for each conjugate β'

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M_σ-invariant decomposition:

$$\mathbb{R}^5 = \mathbb{K}_\beta \oplus \mathbb{K}_n$$

M_σ is hyperbolic on $\mathbb{K}_\beta$.

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M_σ is expanding, contracting on $\mathbb{K}_e$, resp. $\mathbb{K}_c$.

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$$\mathbb{R}^5 \cong \mathbb{R} \oplus \mathbb{C} \oplus \mathbb{K}_n$$

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Substitutive system

For a primitive σ with fixed point u consider

$$X_\sigma = \overline{\{S^k u : k \in \mathbb{Z}\}}$$

where S is the shift.

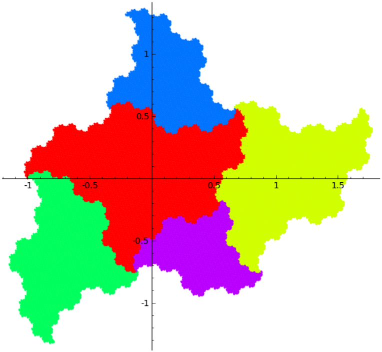
- (X_σ, S) is a subshift.
- (X_σ, S) is minimal, uniquely ergodic, with zero entropy.
- Pisot condition $\Leftrightarrow$ (X_σ, S) not weakly mixing.

Q: Does it have *pure discrete spectrum*?

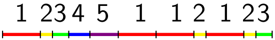
Rauzy fractals

$$\sigma : 1 \mapsto 12,\ 2 \mapsto 3,\ 3 \mapsto 4,\ 4 \mapsto 5,\ 5 \mapsto 1$$

$$u = 1234511212312341234512345112\cdots$$



- Projection of vertices of a broken line.



$$\mathcal{R}(i) = \overline{\{\pi_c \mathbf{l}(p) : pi \text{ prefix of } u\}}$$

- Embedded beta numeration integers:

$$\sum_{k \geq 0} \delta_c(d_k \beta^k),\ (d_k) \leq_{\text{lex}} (1)_\beta$$

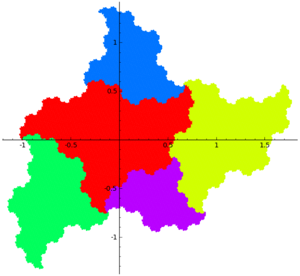
- GIFS directed by the prefix automaton of σ.

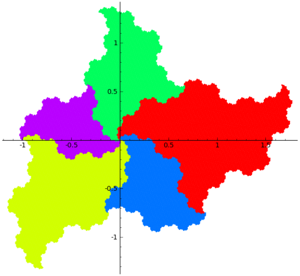
Strong Coincidence Condition

For any $(i,j) \in \mathcal{A}^2$ there is $k \in \mathbb{N}$ such that $\sigma^k(i) \in p_1 a \mathcal{A}^*$ and $\sigma^k(j) \in p_2 a \mathcal{A}^*$ for some $a \in \mathcal{A}$ and words $p_1, p_2 \in \mathcal{A}^*$ with $\mathbf{l}(p_1) = \mathbf{l}(p_2)$.

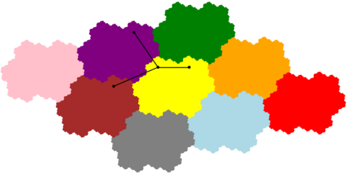
If SCC holds then (X_σ, S) is measurably conjugate to a *domain exchange* $(\mathcal{R}, E)$ defined a.e. by

$$E : \mathbf{x} \mapsto \mathbf{x} + \pi_c(\mathbf{e}_i), \qquad \text{for } \mathbf{x} \in \mathcal{R}(i)$$





Having a periodic tiling by Rauzy fractals implies pure discrete spectrum.
That's true for all irreducible Pisot beta-substitutions (e.g. Tribonacci)!



$$\begin{array}{ccccc} X_\sigma & \xrightarrow{\cong} & \mathcal{R} & \xrightarrow{\cong} & \mathbf{1}^\perp/\Lambda \\ \downarrow S & & \downarrow E & & \downarrow E \\ X_\sigma & \longrightarrow & \mathcal{R} & \longrightarrow & \mathbf{1}^\perp/\Lambda \end{array}$$

- $\mathbf{1}^\perp$ is the hyperplane orthogonal to $(1, \ldots, 1)$
- $\Lambda = \sum_{i \in \mathcal{A}} \mathbb{Z}(\mathbf{e}_i - \mathbf{e}_1)$

Periodic tilings

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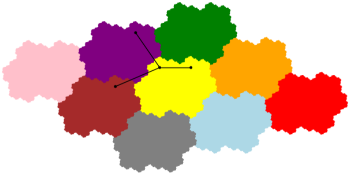

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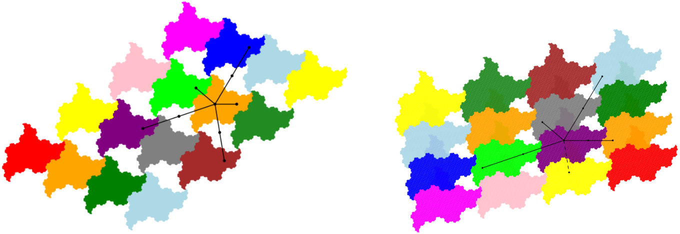
Existence of necessary and sufficient geometrical, combinatorial and arithmetical conditions to get tilings.

The Pisot Conjecture

But. . . unfortunately the Hokkaido Rauzy fractal does not tile periodically $\mathbb{C}$!



Reducible $\to$ linear dependencies

$$\pi(\mathbf{e}_1) = \pi(\mathbf{e}_3) + \pi(\mathbf{e}_4),$$
$$\pi(\mathbf{e}_5) = \pi(\mathbf{e}_2) + \pi(\mathbf{e}_3).$$

Combinatorially

$\chi : 1 \mapsto 34, 2 \mapsto 2, 3 \mapsto 3, 4 \mapsto 4, 5 \mapsto 32.$

Broken lines

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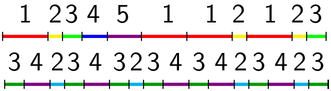

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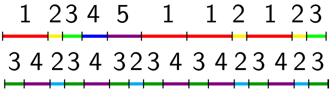

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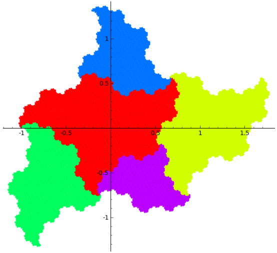
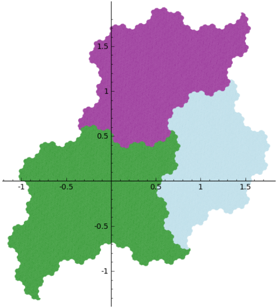
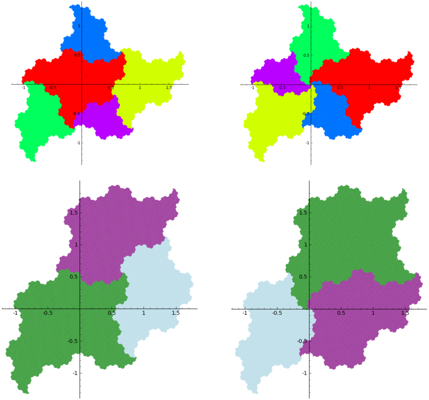




Domain exchange



- $(\mathcal{R}, E)$ domain exchange on the original Hokkaido tile.

$$E : \mathcal{R}(i) \mapsto \mathcal{R}(i) + \pi_c(\mathbf{e}_i), \ i \in \mathcal{A}$$

- $(\widetilde{\mathcal{R}}, \widetilde{E})$ is a *toral translation*, since it induces a periodic tiling of $\mathbb{C}$ mod $\Lambda = \pi_c((\mathbf{e}_4 - \mathbf{e}_3)\mathbb{Z} + (\mathbf{e}_4 - \mathbf{e}_2)\mathbb{Z})$.

$$\widetilde{E} : \widetilde{\mathcal{R}}(i) \mapsto \widetilde{\mathcal{R}}(i) + \pi_c(\mathbf{e}_i), \ i \in \{2, 3, 4\}$$

- E is the *first return* of $\widetilde{E}$ on $\mathcal{R}$.

Codings of the domain exchange

Let $\Omega = \overline{\{S^k w : k \in \mathbb{N}\}}$, where $w = \chi(u)$ is the coded fixed point of σ.

We have the following commutative diagram:

$$
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X_\sigma & \xrightarrow{\chi} & \Omega & \xrightarrow{\phi} & \widetilde{\mathcal{R}} & \longrightarrow & \mathbb{C}/\Lambda \\
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We can generalize what shown for the family of substitutions

$$\sigma_t : \ 1 \mapsto 1^{t+1}2, \ 2 \mapsto 3, \ 3 \mapsto 4, \ 4 \mapsto 1^t 5, \ 5 \mapsto 1$$

(Arnoux, Ito 01) formalism for *irreducible* Pisot substitutions.

Action of the substitution on 1-dimensional faces $\rightarrow$ broken line
For $(\mathbf{x}, a) \in \mathbb{Z}^d \times \mathcal{A}$

$$\mathbf{E}_1(\sigma)(\mathbf{x}, a) = \sum_{\sigma(a)=pbs} (M_\sigma \mathbf{x} + \mathbf{l}(p), b)$$

Dual action on $(d-1)$-dimensional faces:

$$\mathbf{E}_1^*(\sigma)(\mathbf{x}, a)^* = \sum_{\sigma(b)=pas} (M_\sigma^{-1}(\mathbf{x} - \mathbf{l}(p)), b)^*$$

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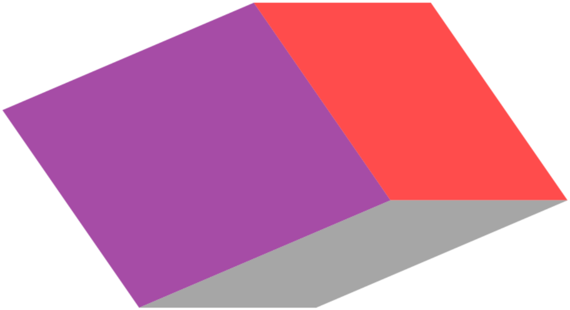
Useful for:

1. generating Rauzy fractals as *Hausdorff limits*.
2. producing *stepped surfaces*.

$$\mathcal{R}(a) = \lim_{k \to \infty} \pi_c(M_\sigma^k \, \mathbf{E}_1^*(\sigma)^k(\mathbf{0}, a)^*)$$



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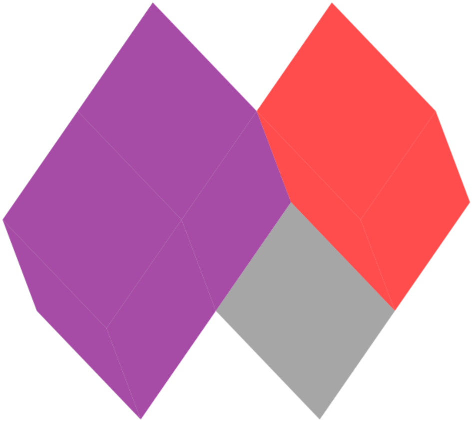

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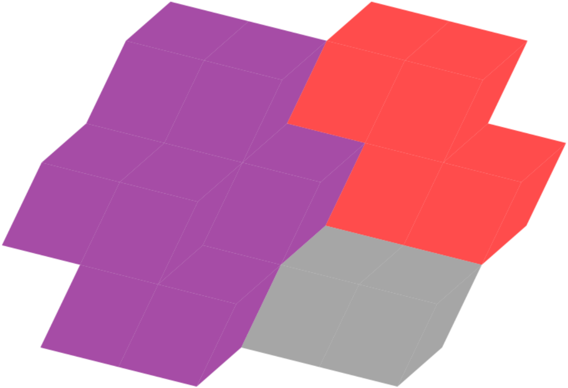

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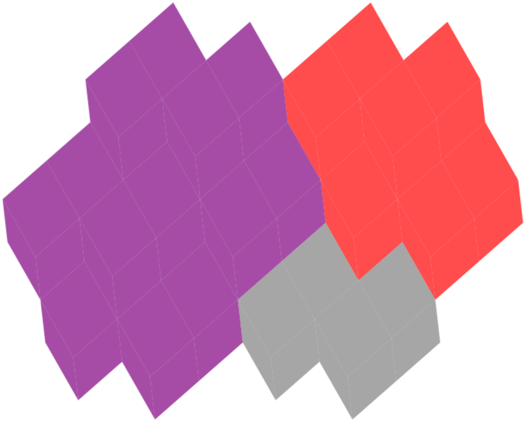

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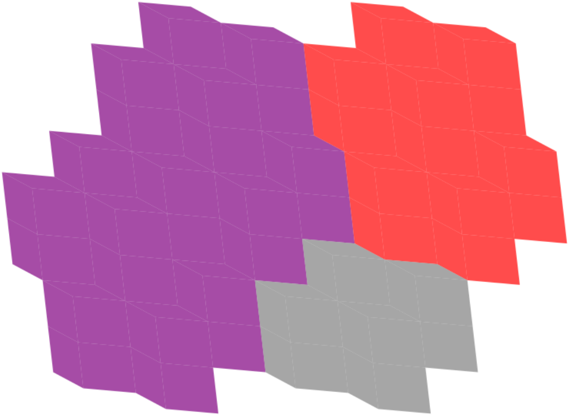

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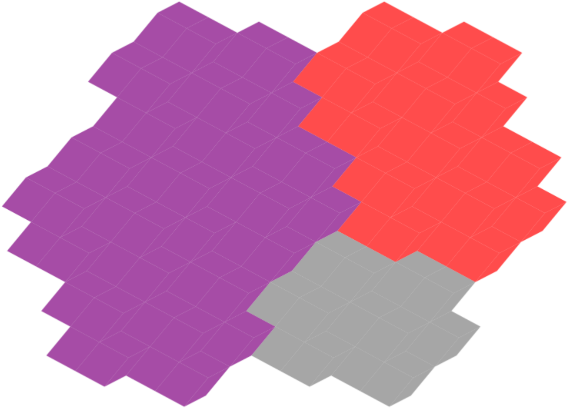

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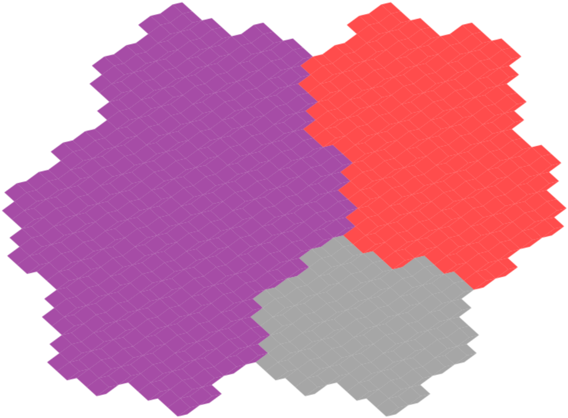

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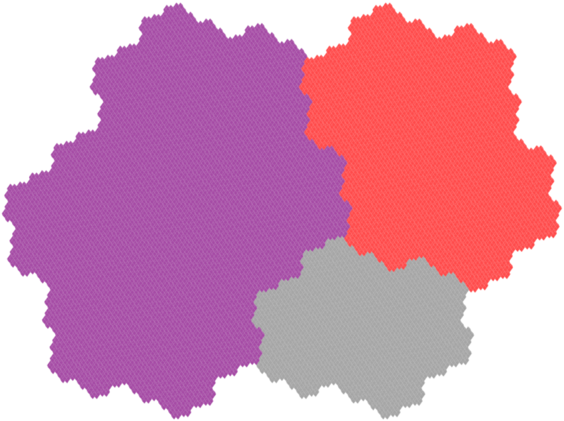

Set of coloured points "near" $\mathbb{K}_c$:

$$\Gamma = \{(\mathbf{x}, a) \in \mathbb{Z}^d \times \mathcal{A} : \mathbf{x} \in (\mathbb{K}_c)^{\geq}, \mathbf{x} - \mathbf{e}_a \in (\mathbb{K}_c)^{<}\}$$

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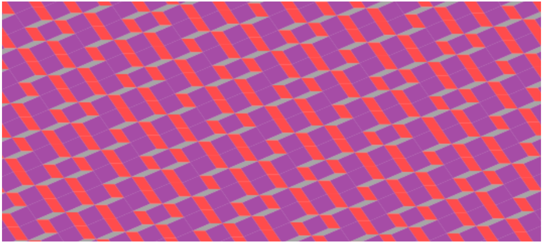
- $\mathbf{E}_1^*(\sigma)(\Gamma) = \Gamma \ \rightarrow$ self-replicating property (Kenyon).
- Aperiodic translation set (Delone set) for a self-replicating multiple tiling made of Rauzy fractals.
- Geometric representation as an arithmetic discrete model of the hyperplane $\mathbb{K}_c$, whose projection is a polygonal tiling.



Reducible case: $\boxed{n = \#\mathcal{A} > d = \deg(\beta)}$.

Higher dimensional dual maps

Reducible case: $\boxed{n = \#\mathcal{A} > d = \deg(\beta)}$.

We want to work with $(d-1)$-dimensional faces! The dual map $\mathbf{E}^*_{n-d+1}(\sigma)$ will suit:

$$\mathbf{E}^*_{n-d+1}(\sigma)(\mathbf{x}, \underline{a})^* = \sum_{\underline{b} \xrightarrow{\underline{p}} \underline{a}} \left(M_\sigma^{-1}(\mathbf{x} - \mathbf{l}(\underline{p})), \underline{b}\right)^*$$

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Remarks:

- $\mathbf{E}^*_{n-d+1}(\sigma)$ acts on $\binom{n}{n-d+1}$ oriented faces.
- If σ is irreducible $n = d$ and $\mathbf{E}^*_{n-d+1}(\sigma) = \mathbf{E}^*_1(\sigma)$.
- $\mathbf{E}_k(\sigma)$ and $\mathbf{E}^*_k(\sigma)$ commute in general with boundary and coboundary operators (Sano, Arnoux, Ito 2001).
- Similar approach for the study of a free group automorphism associated with a complex Pisot root (Arnoux, Furukado, Harriss, Ito 2011).

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1. Geometric representation for stepped surfaces.
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3. Periodic (multiple) tiling.

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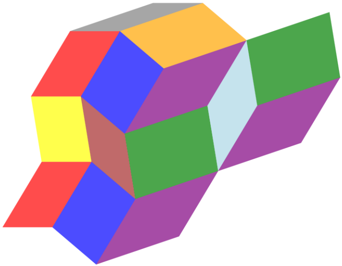

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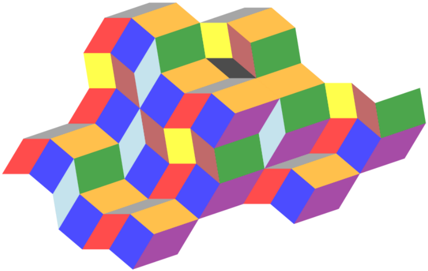

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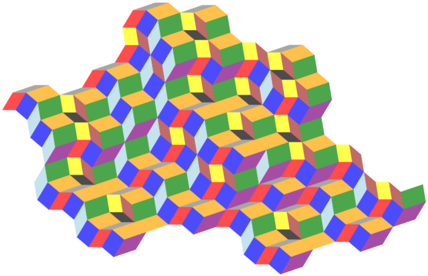

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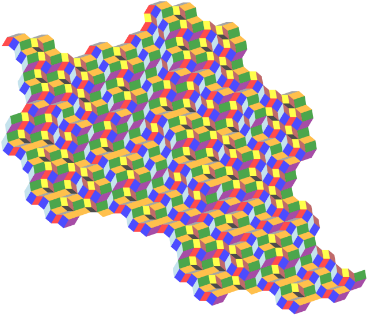

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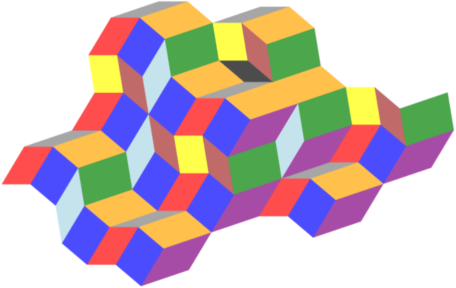

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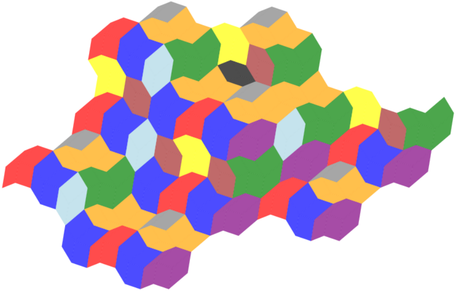

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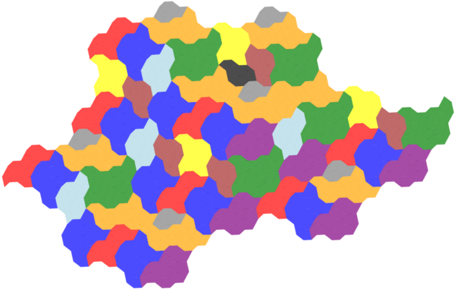

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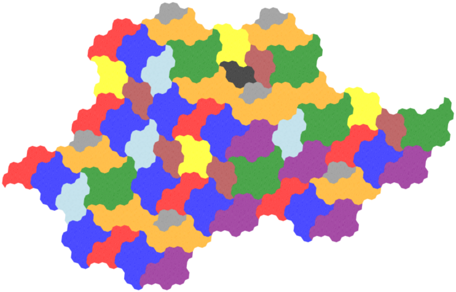

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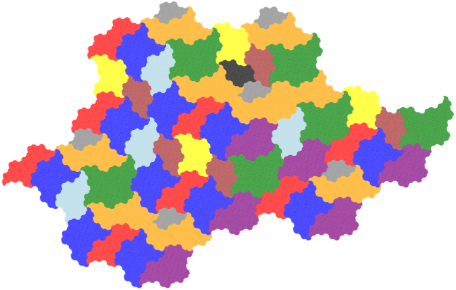

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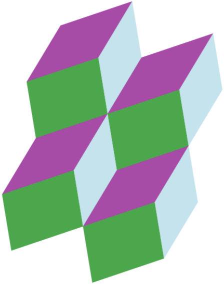

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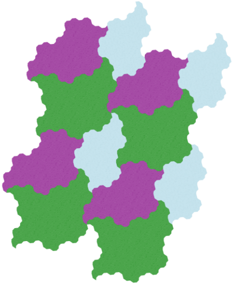

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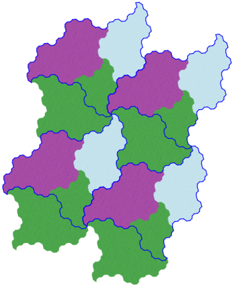
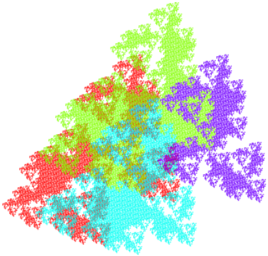


(Joint work with X. Bressaud and T. Jolivet)

The geometrical interpretation seems to get harder for these substitutions, not satisfying the strong coincidence condition:

$$\sigma : 1 \mapsto 14, 2 \mapsto 32, 3 \mapsto 21, 4 \mapsto 3$$

$$\mathrm{char}(M_\sigma) = (x-1)(x^3 - x^2 - x - 1)$$



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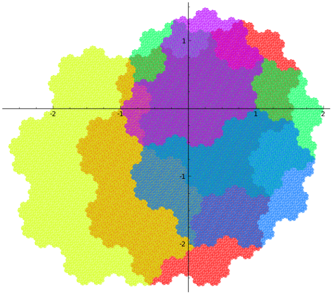
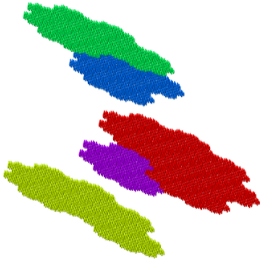
$$\sigma : 1 \mapsto 213, 2 \mapsto 4, 3 \mapsto 5, 4 \mapsto 1, 5 \mapsto 21$$

$$\mathrm{char}(M_\sigma) = (x^2 + x + 1)(x^3 - 2x^2 + x - 1)$$



Projection: $\pi_{c,n} : \mathbb{R}^n \to \mathbb{K}_c \oplus \mathbb{K}_n$.



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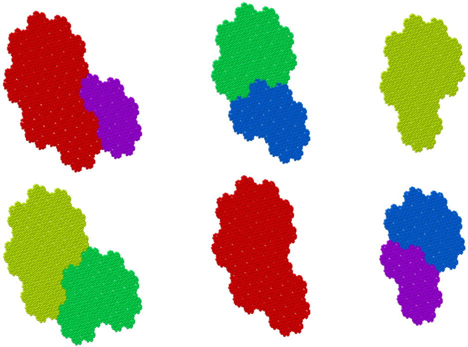

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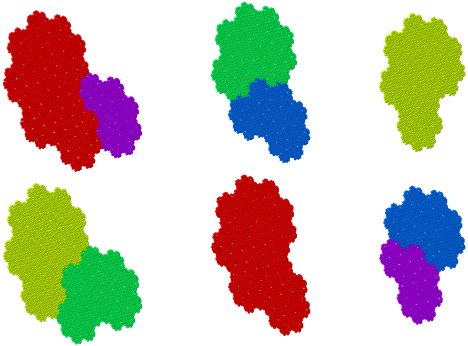
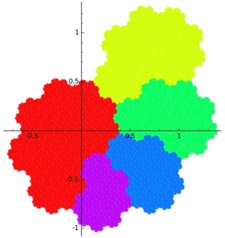


Criterion (depending on the prefix automaton of the substitution) to know whether we get only *finitely many* layers.

In this case $\rightarrow$ NEW strong coincidence condition.

Projecting down suitably we can glue the subtiles together…



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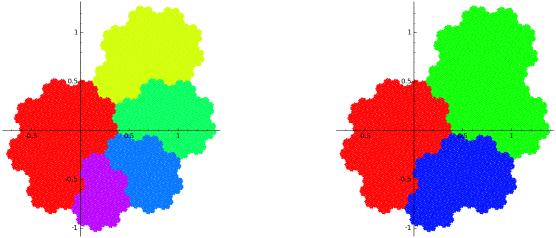

Figure : Symbolic splitting associated with the irreducible substitution $\tau : 1 \mapsto 12, 2 \mapsto 32, 3 \mapsto 1$.

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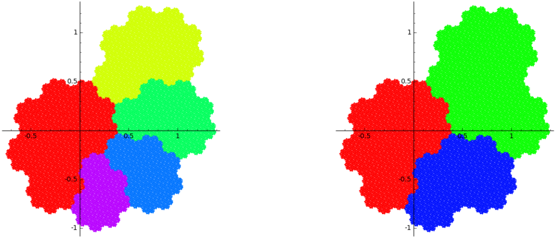


Figure : Symbolic splitting associated with the irreducible substitution $\tau : 1 \mapsto 12, 2 \mapsto 32, 3 \mapsto 1$.

...and obtain the connection with an irreducible substitution.

Philosophy: Dynamically the reducible substitutive system behaves exactly as the irreducible one, after identifying some letters / changing projection. Technique: symbolic splitting.

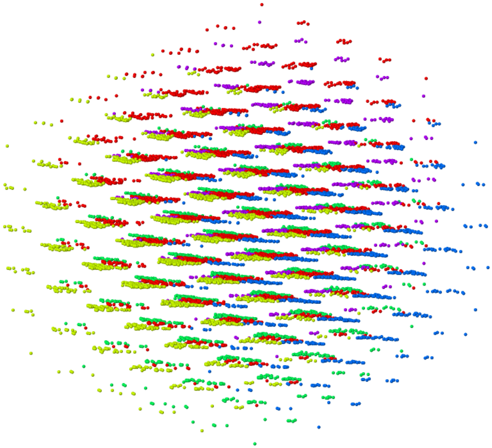
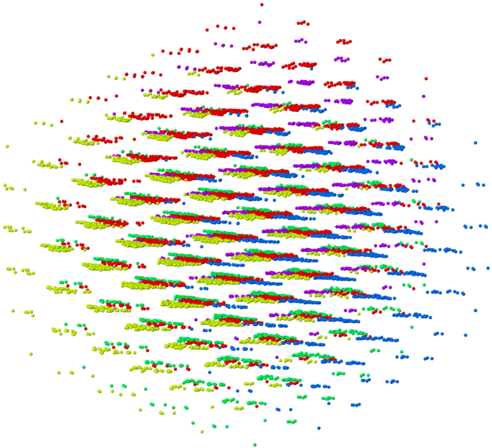


Figure : Rauzy fractal of the Hokkaido substitution in $\mathbb{K}_c \oplus \mathbb{K}_n$. The points distribute with logarithmic growth on a two-dimensional lattice.

Thank you!



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