Finite automata algorithms:

bisimulations up to congruence, and binary decision diagrams

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Journées SDA2, Marne la Vallée, o8.o4.2o15

Language equivalence of automata

- Model checking
- Program equivalence (cf. D'Antoni's work)
- SDN analysis (cf. NetKAT)
- Formal proof automation (Relational methods, Kleene algebra)

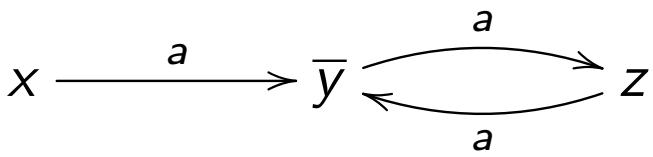
Outline

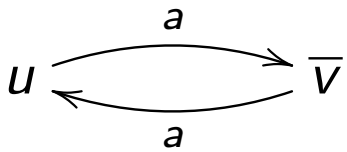
Many states

Many letters

Checking language equivalence of finite automata

Deterministic case, naive algorithm:





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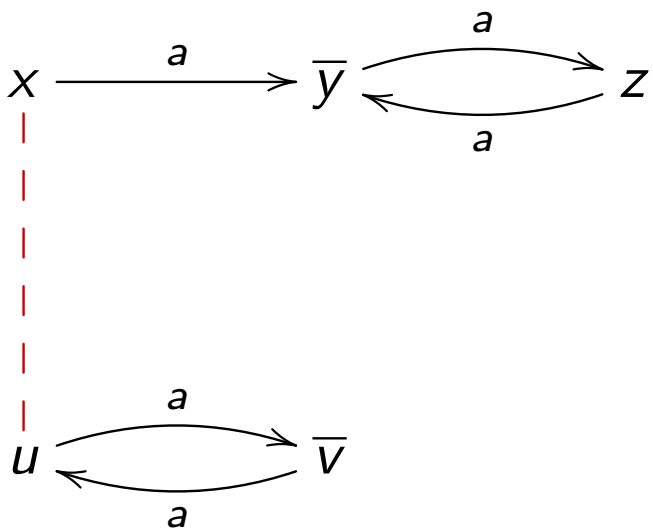

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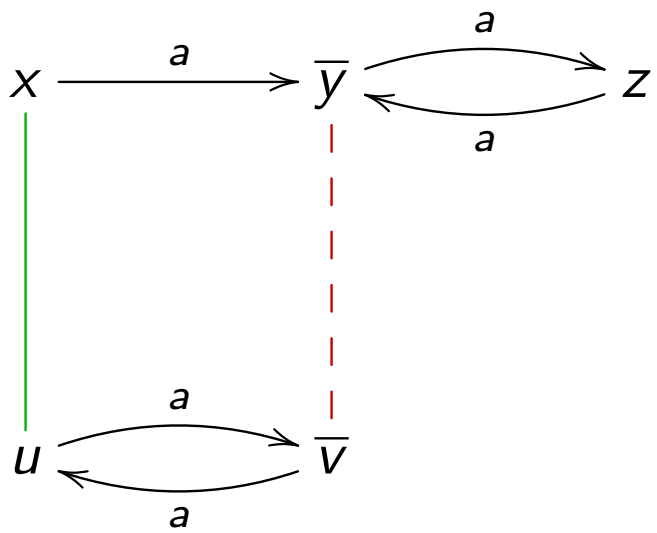

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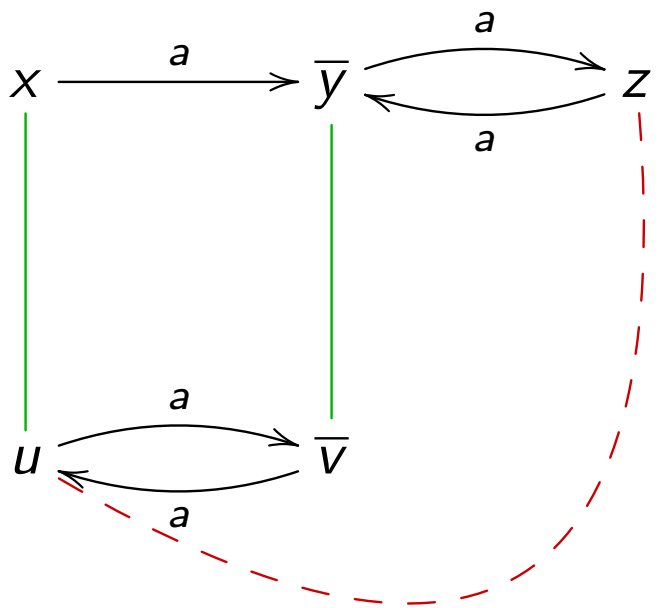

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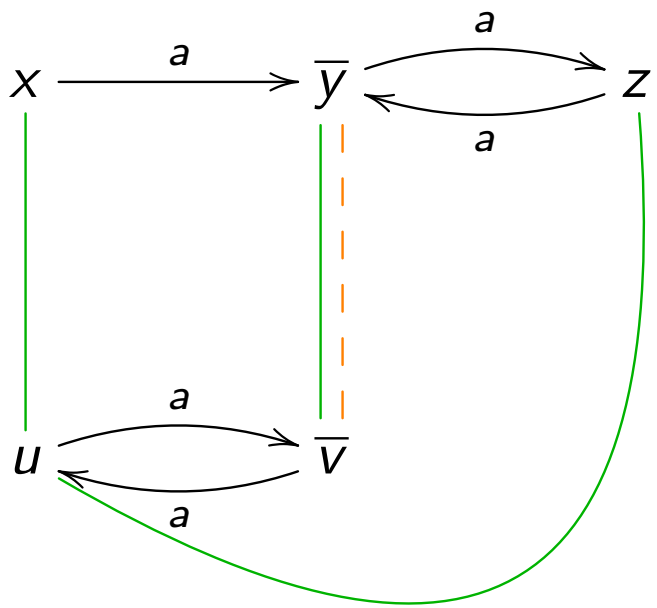

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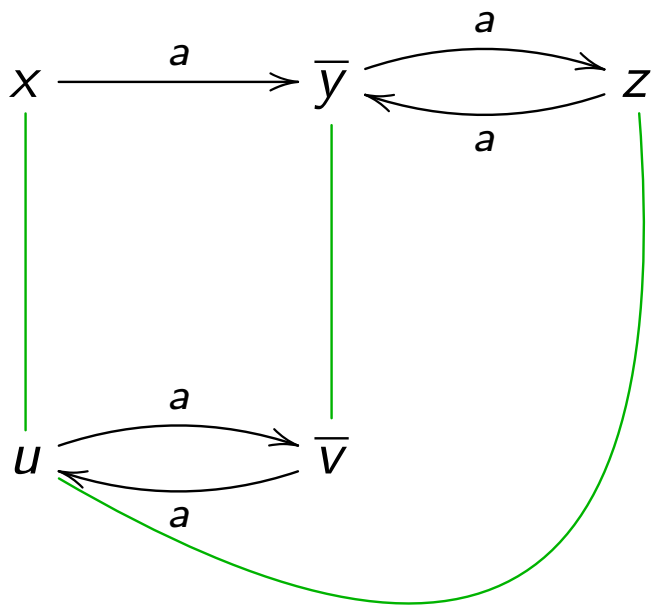

Checking language equivalence

Deterministic case, naive algorithm, correctness:

bisimulation

Theorem: $x \sim y$ iff there exists a bisimulation R with $x \, R \, y$

Checking language equivalence

Deterministic case, naive algorithm, correctness:

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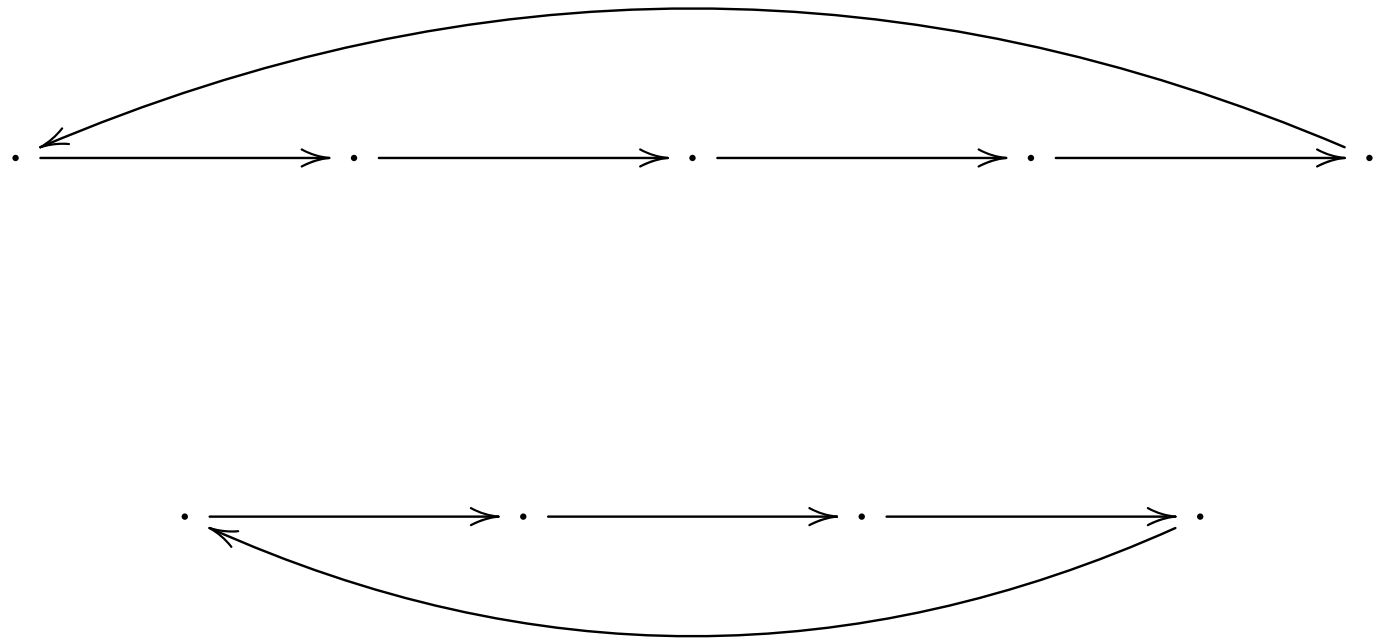
Theorem: $x \sim y$ iff there exists a bisimulation R with $x \, R \, y$

The previous algorithm attempts to construct a bisimulation

Checking language equivalence

Deterministic case, naive algorithm: quadratic complexity



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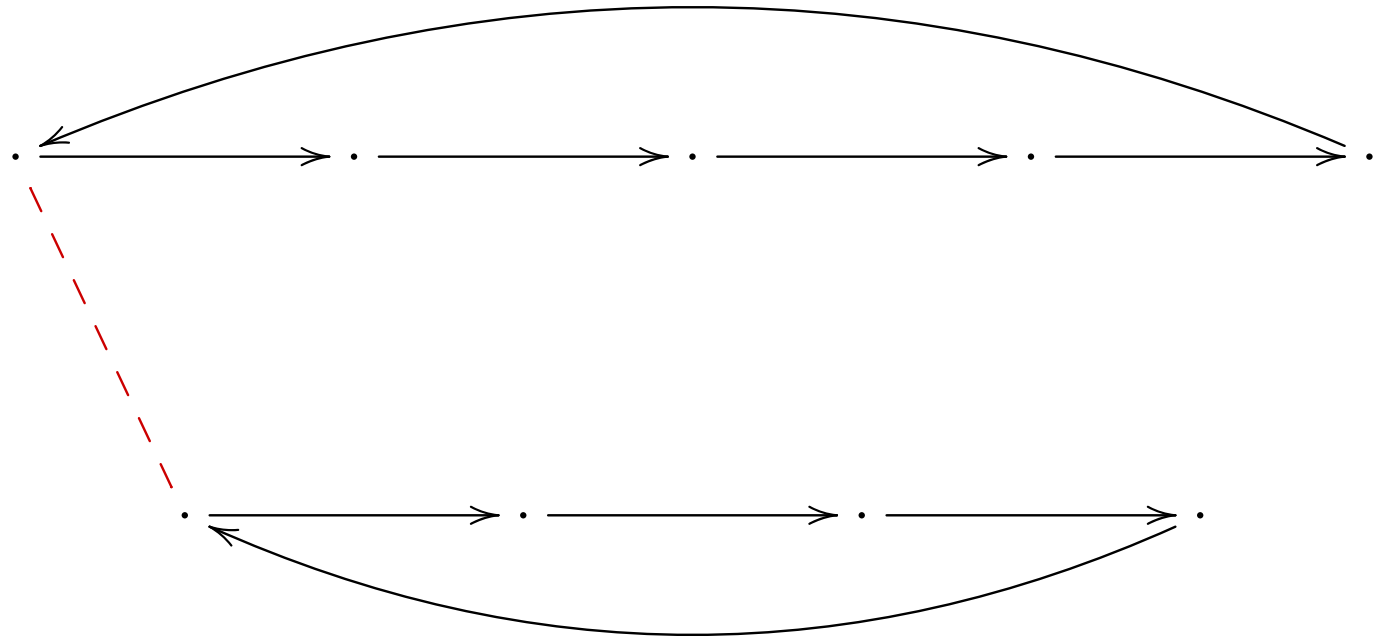
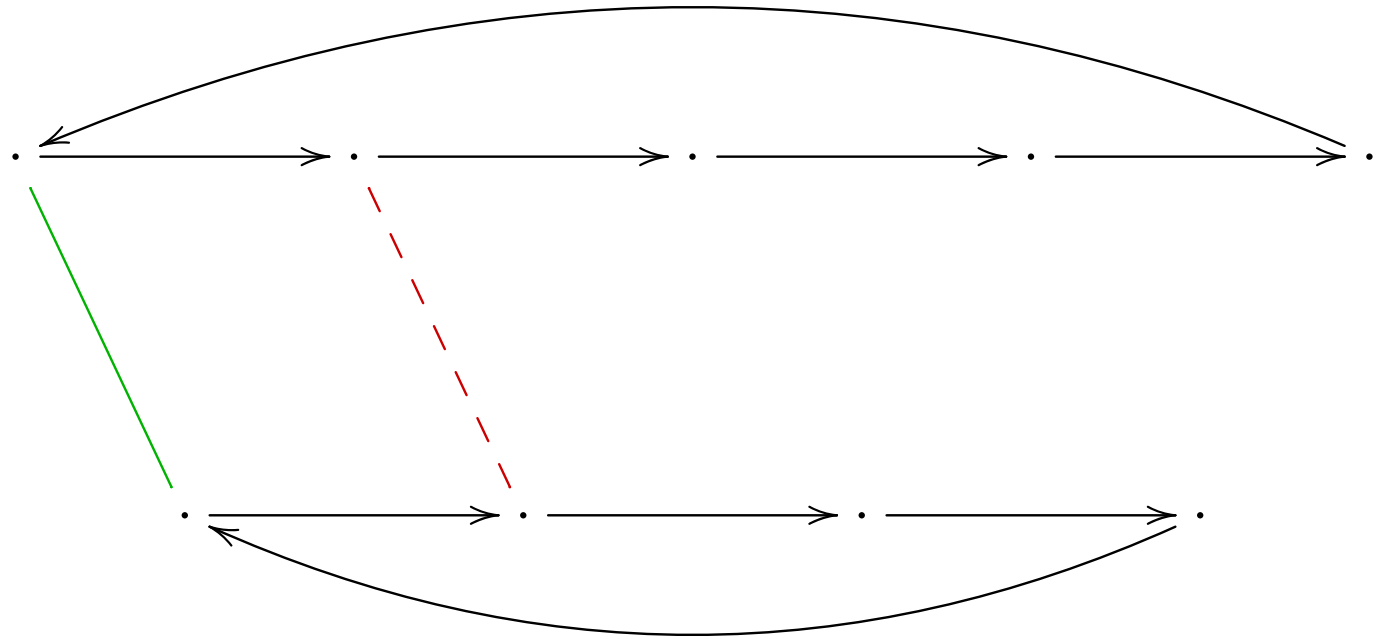


1 pairs

Checking language equivalence

Deterministic case, naive algorithm: quadratic complexity



2 pairs

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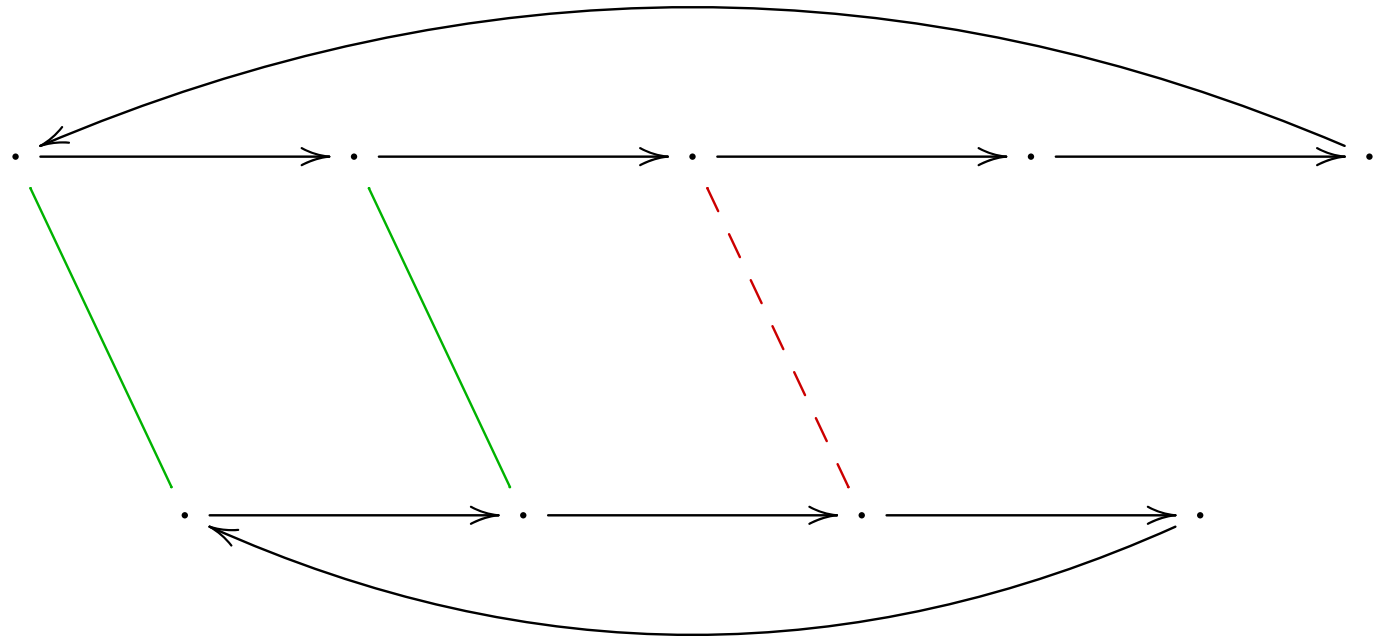

3 pairs

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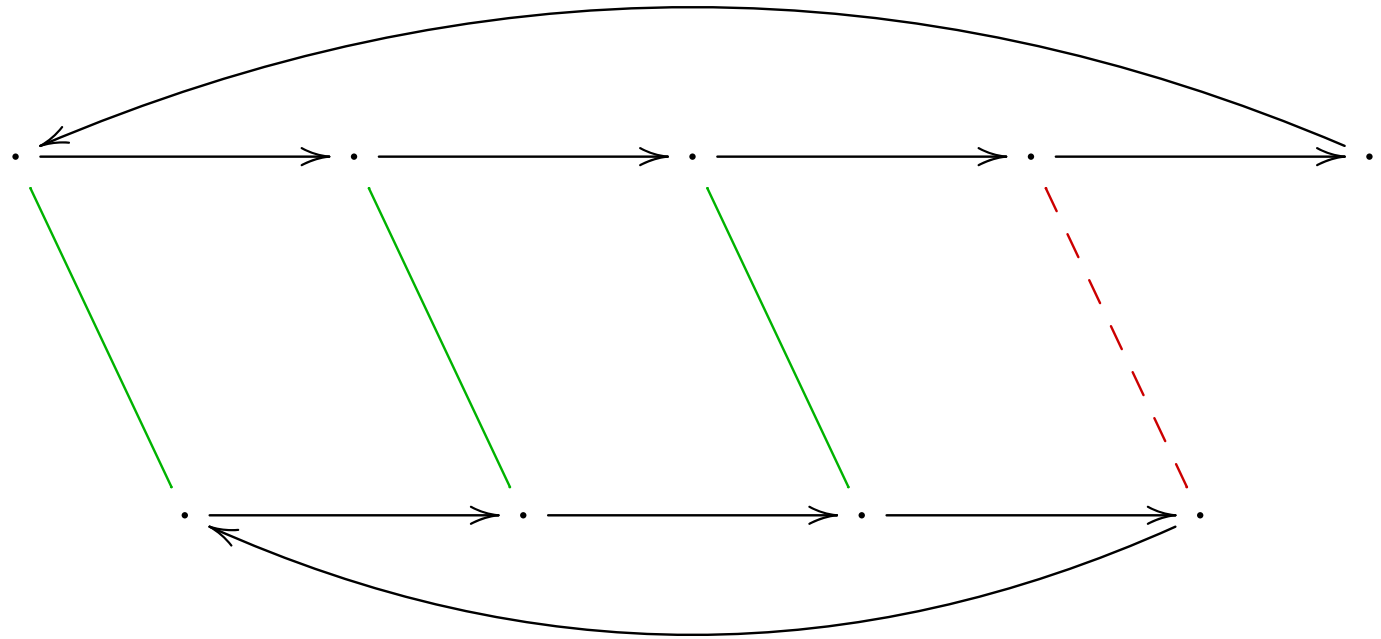

4 pairs

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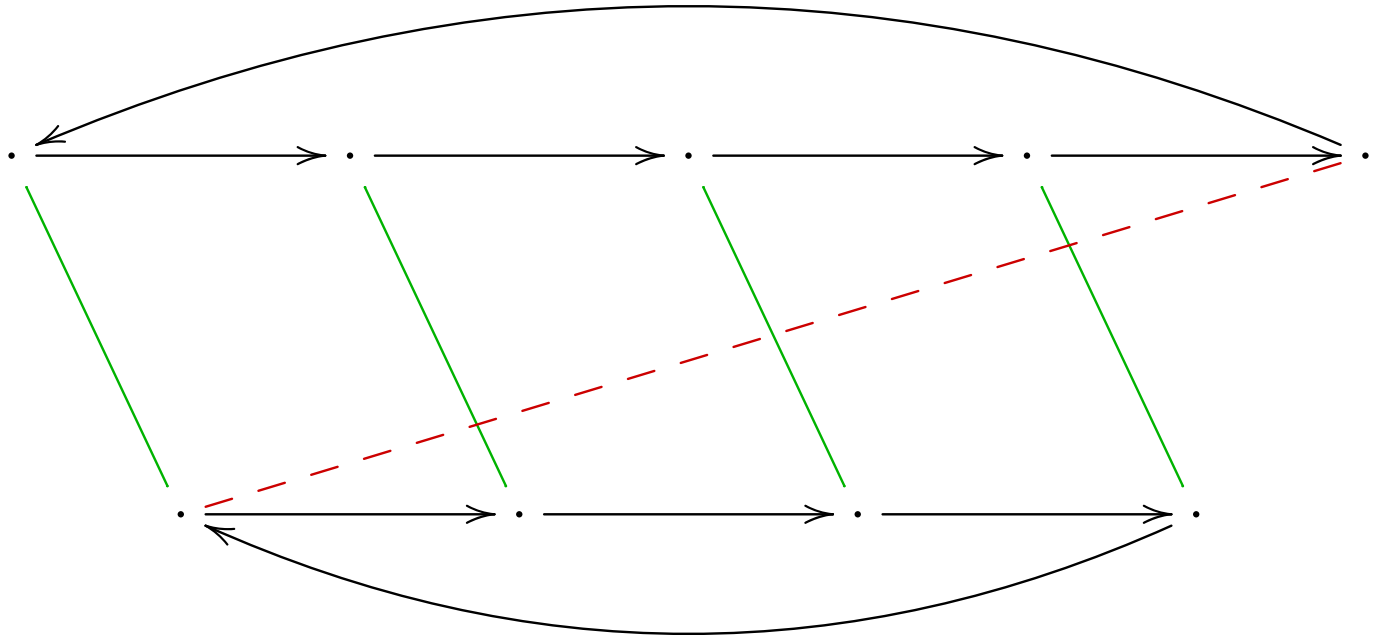

5 pairs

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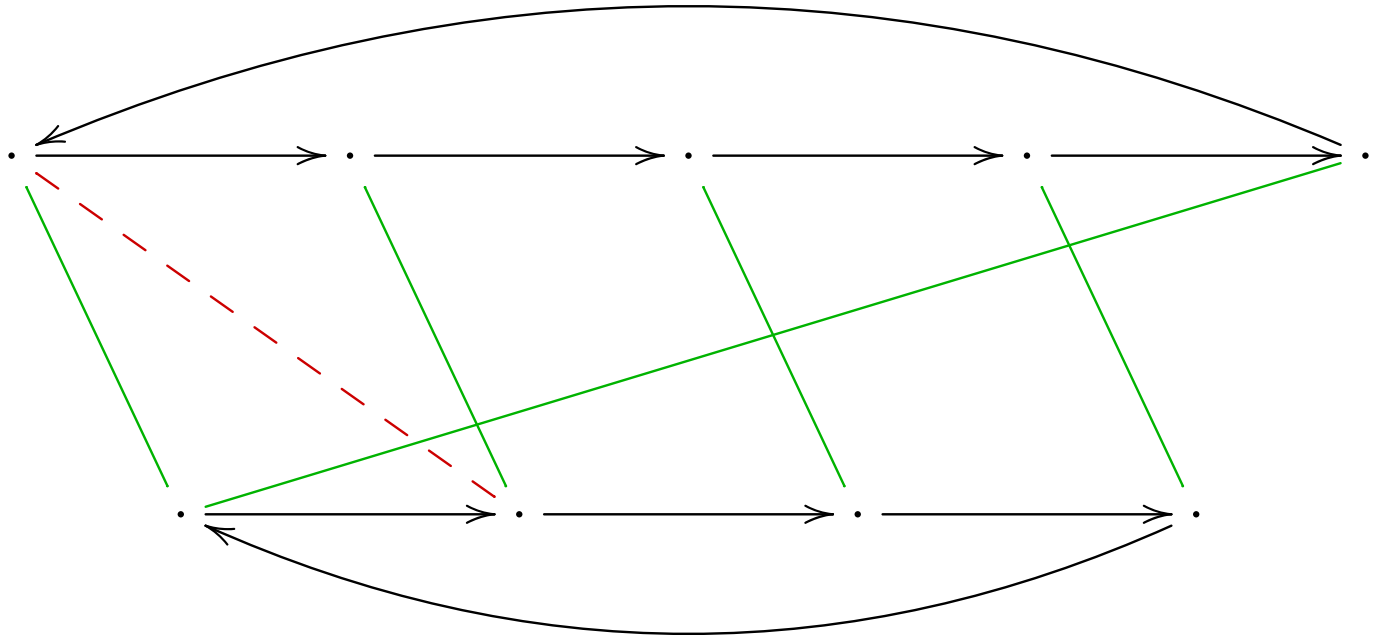

6 pairs

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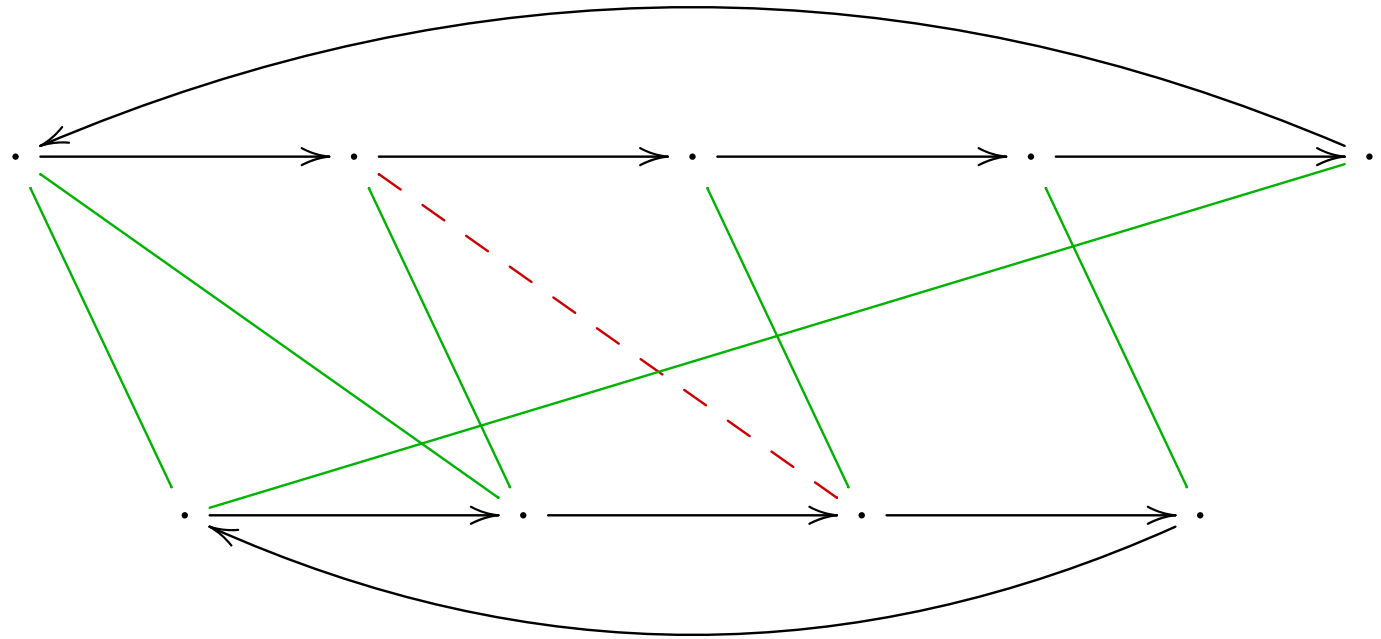

7 pairs

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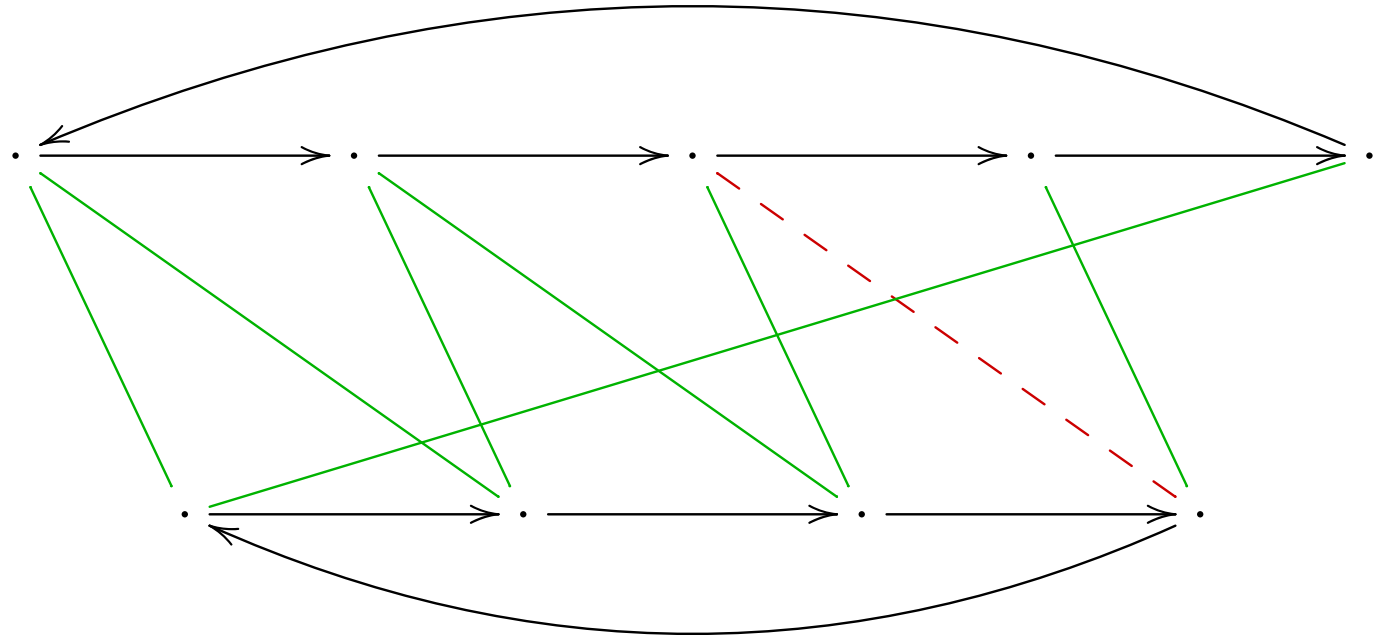

8 pairs

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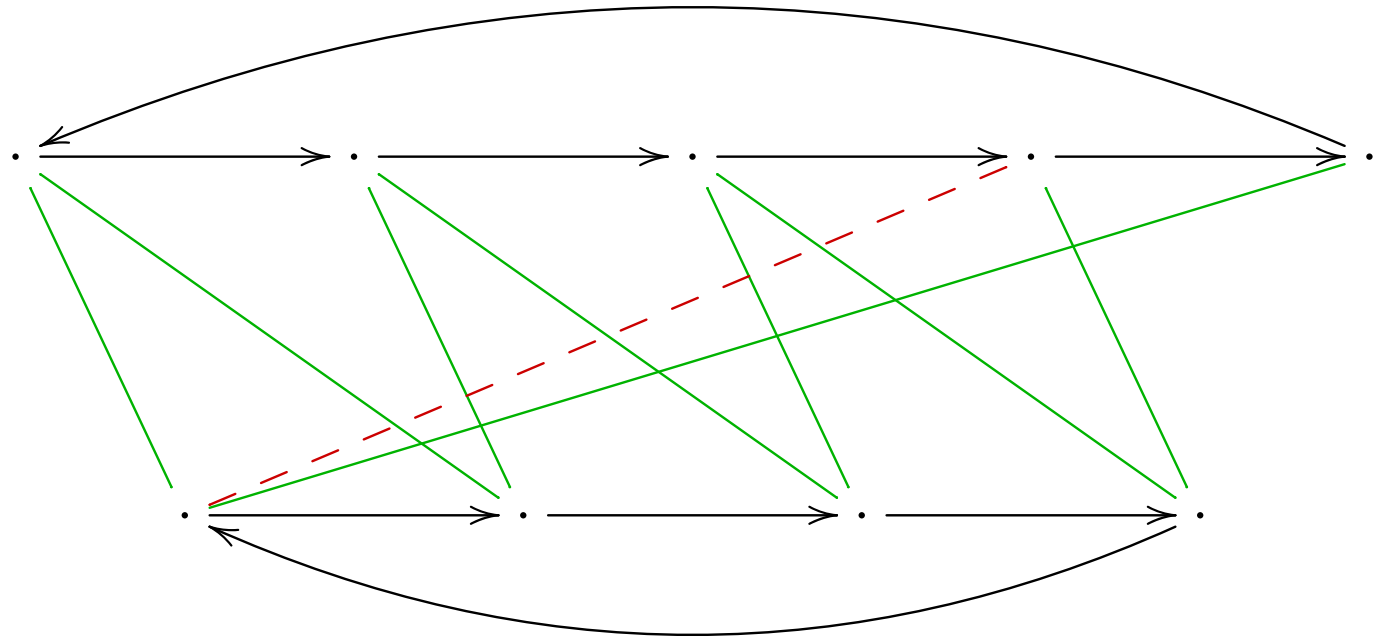

9 pairs

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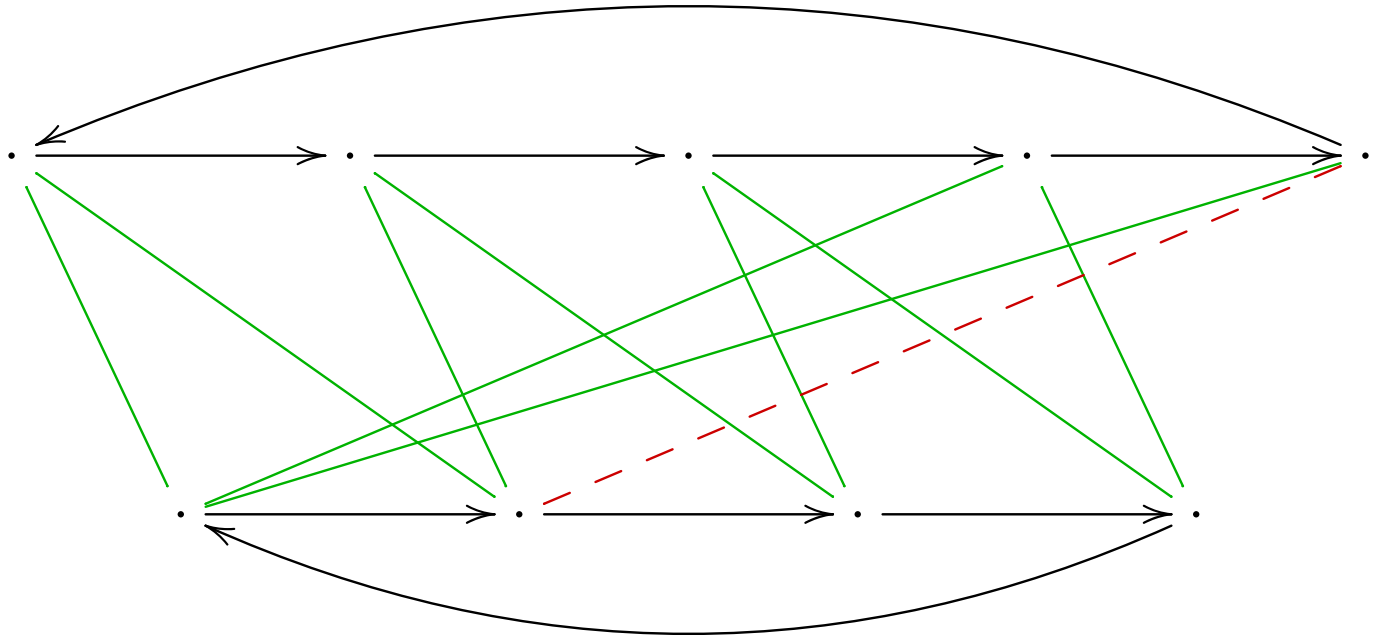

10 pairs

Checking language equivalence

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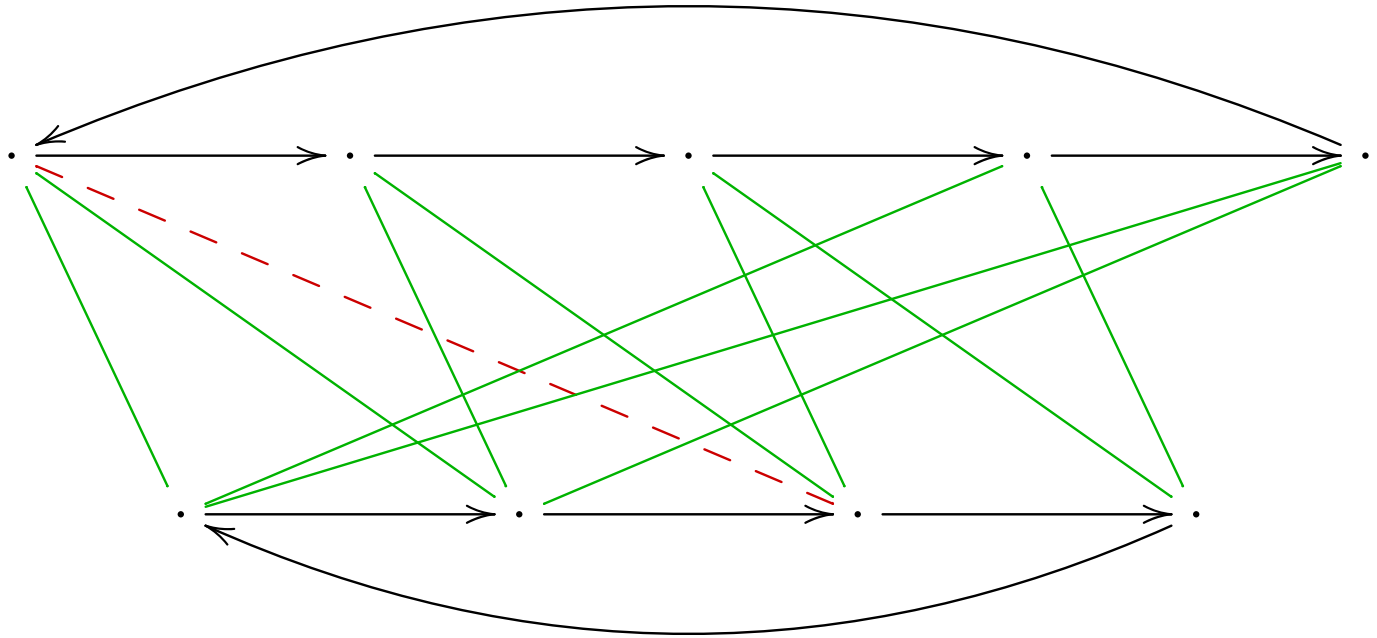

11 pairs

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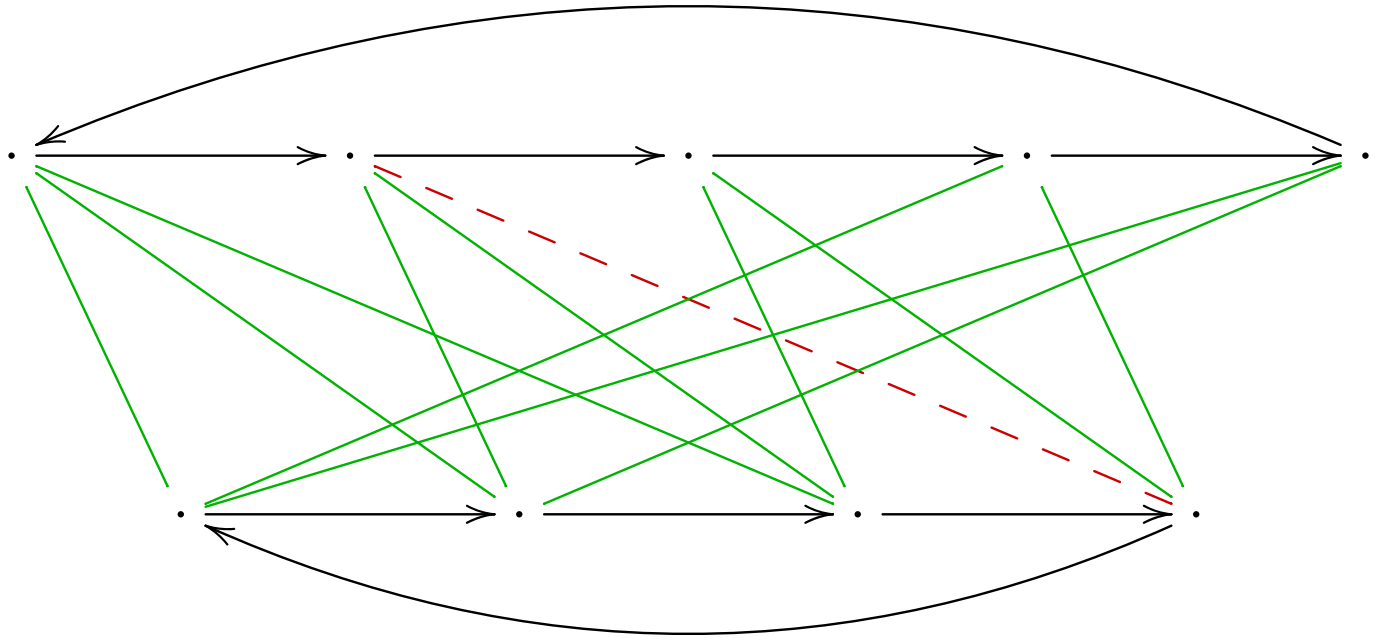

12 pairs

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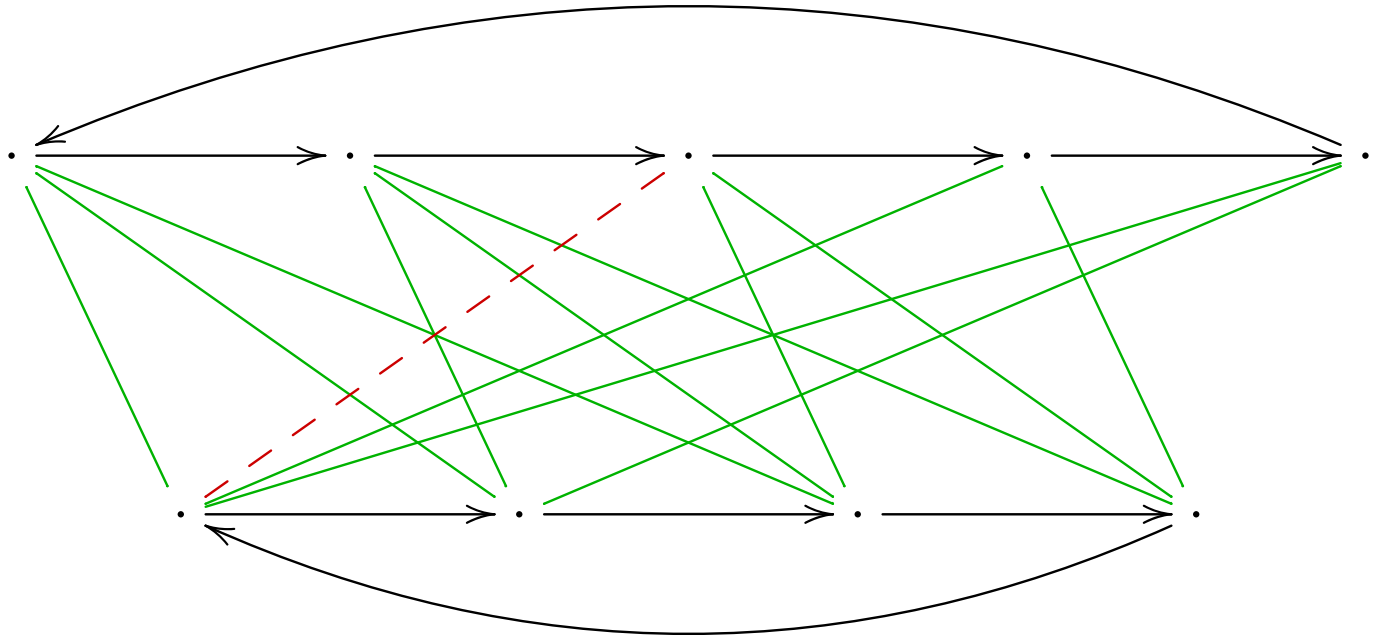

13 pairs

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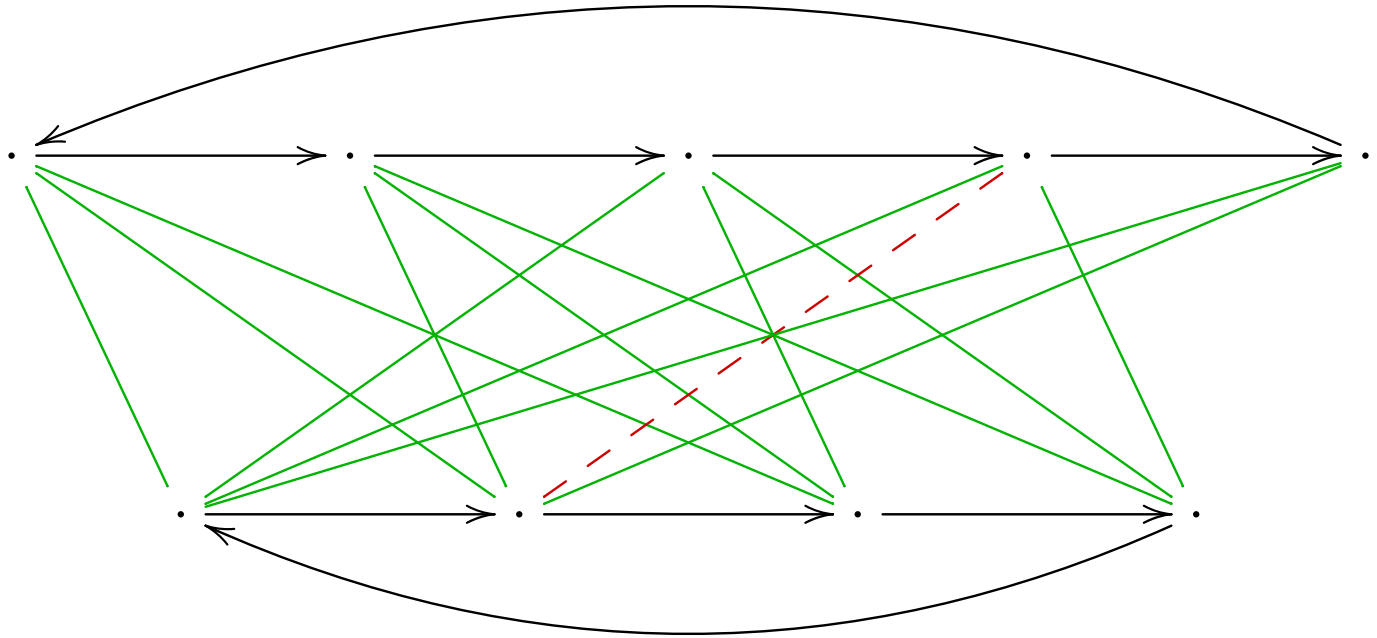

14 pairs

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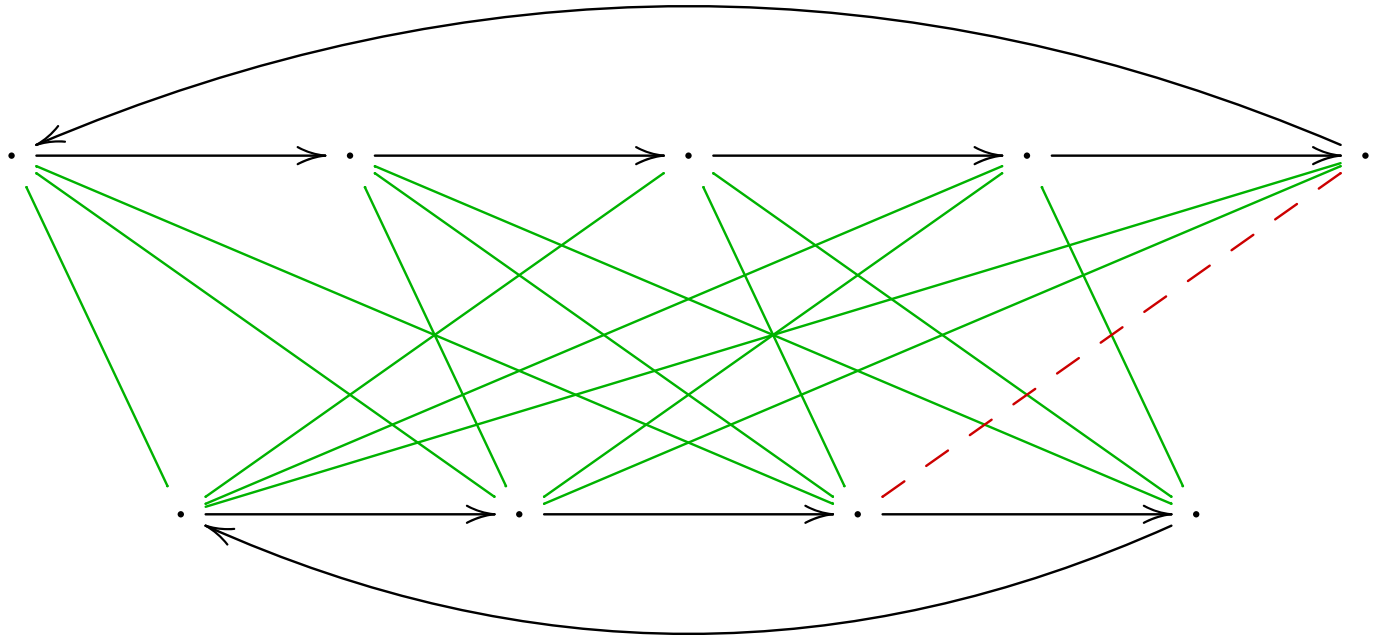

15 pairs

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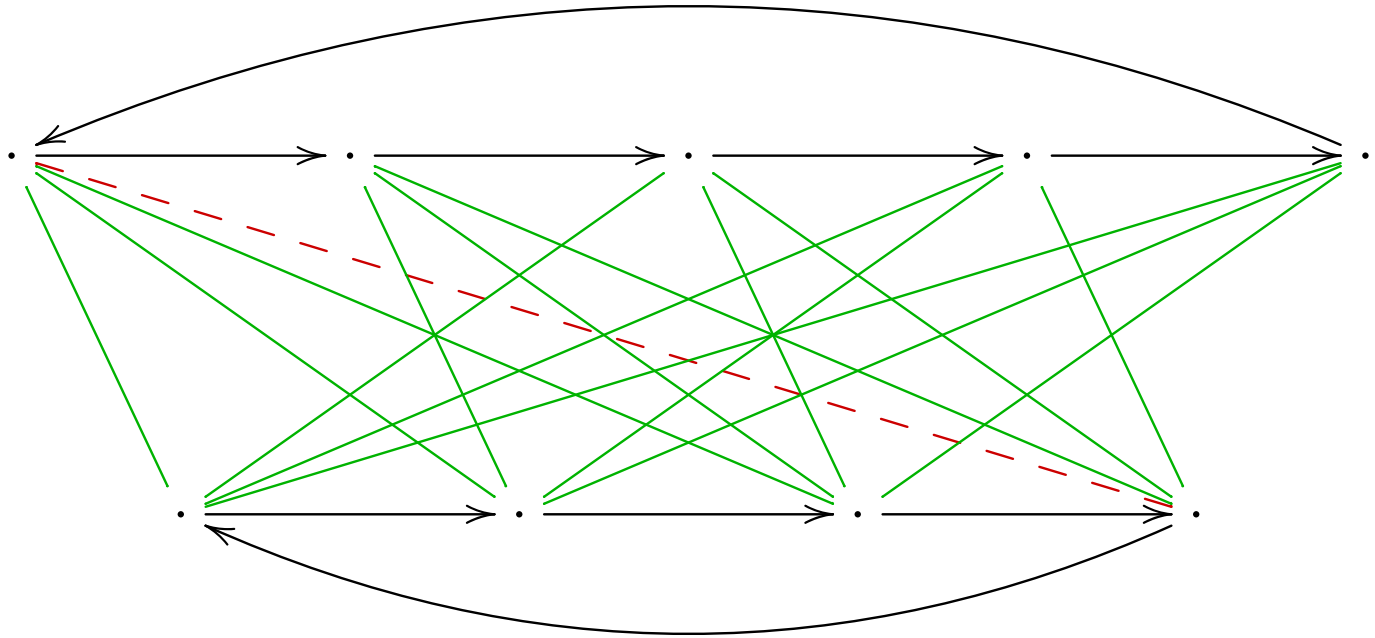

16 pairs

Checking language equivalence

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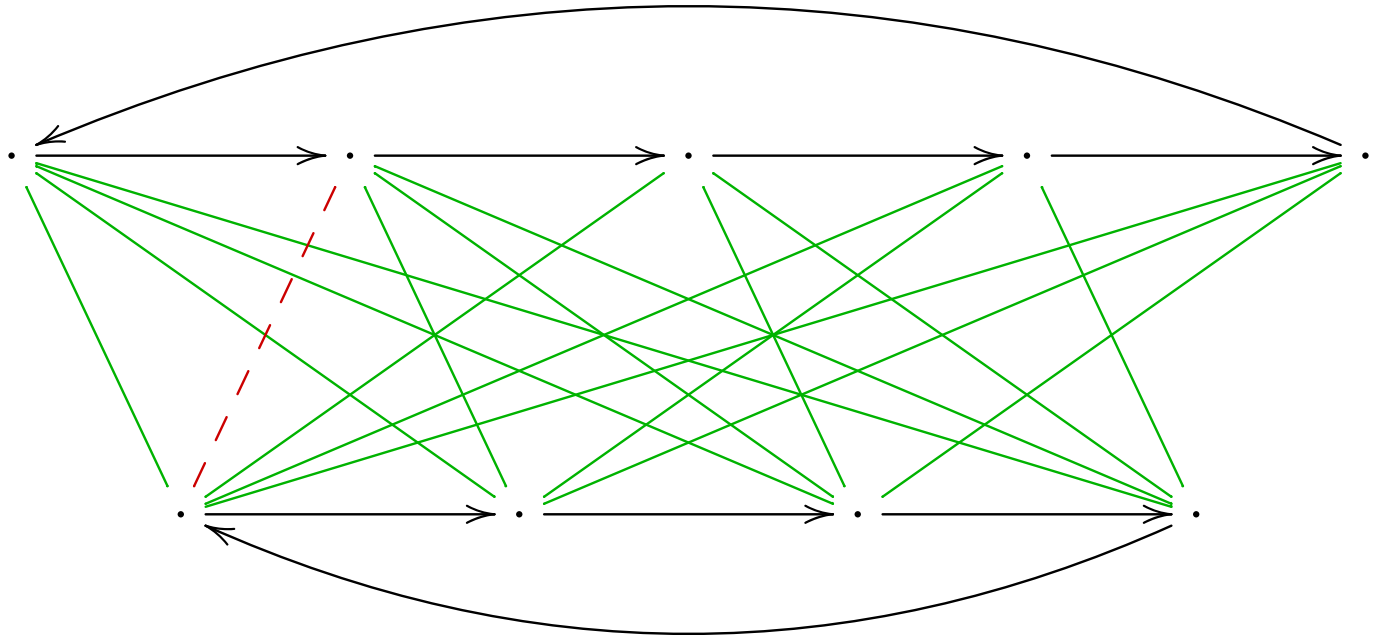

17 pairs

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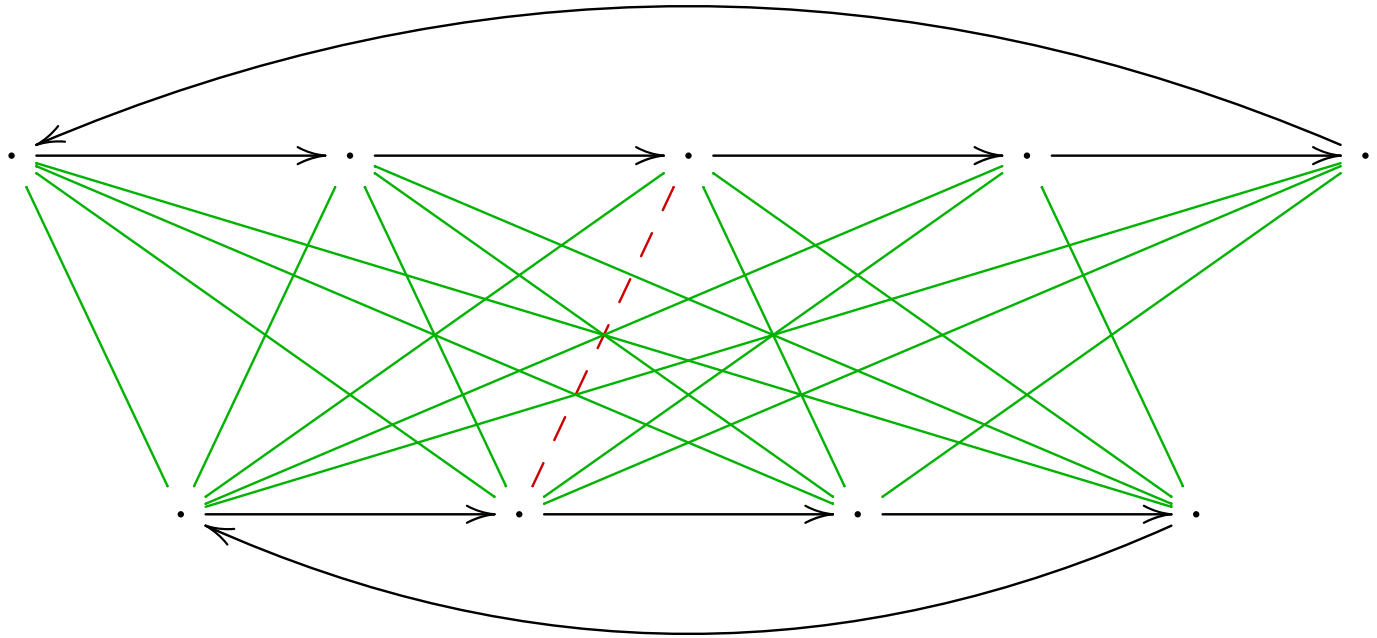
Deterministic case, naive algorithm: quadratic complexity



18 pairs

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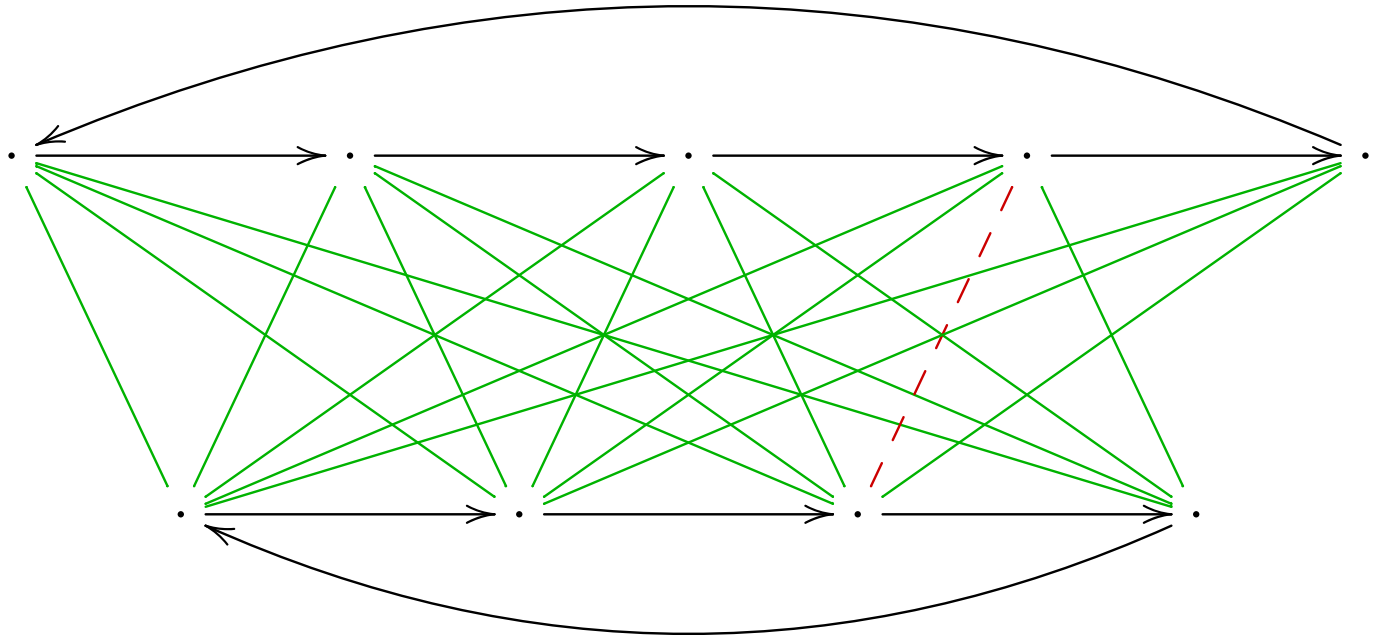

19 pairs

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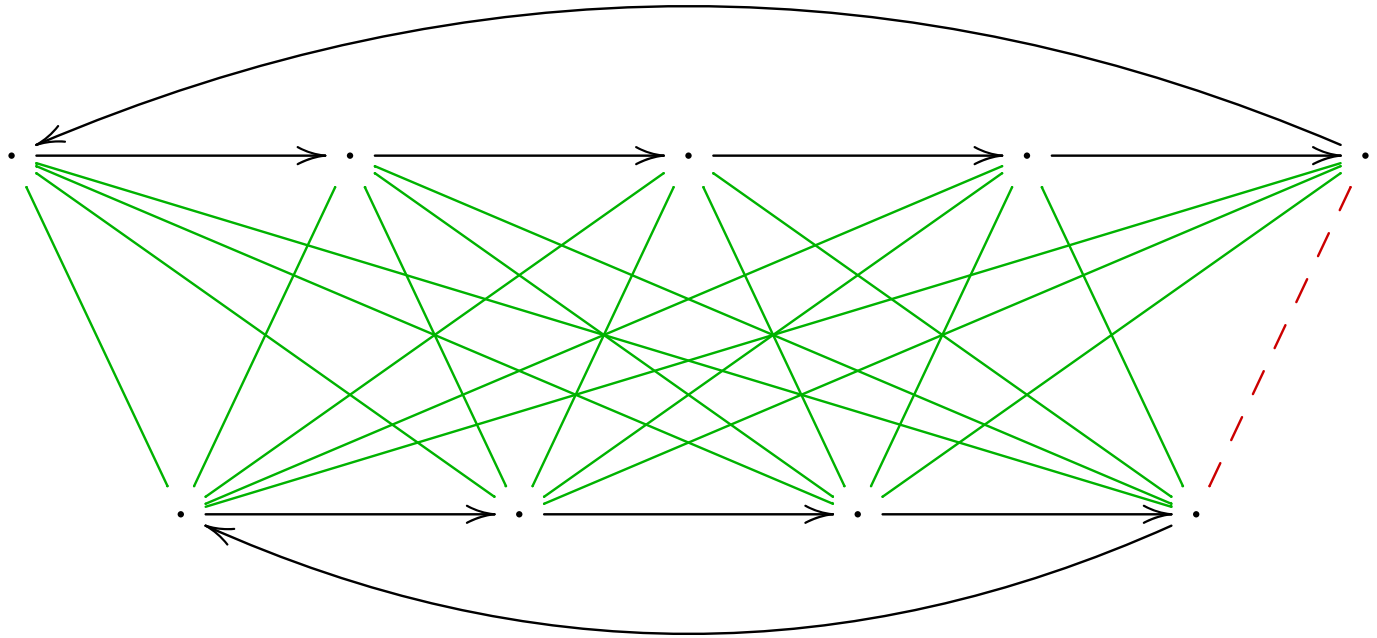

20 pairs

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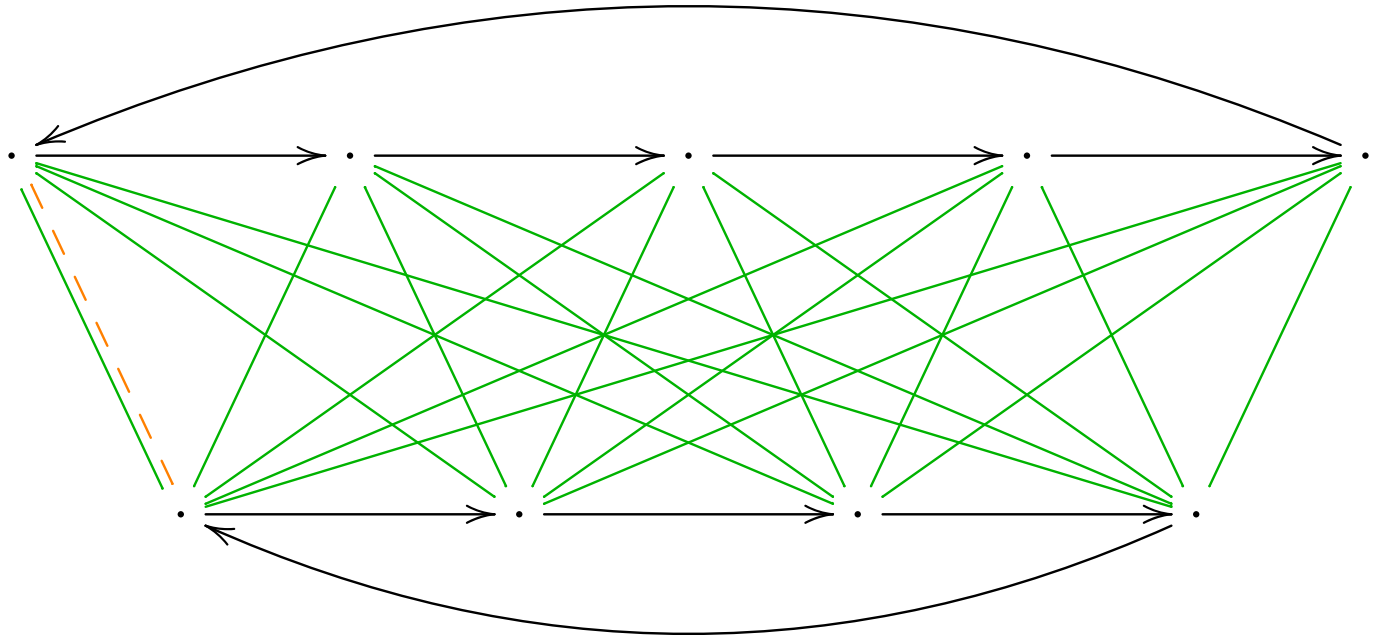

20 pairs

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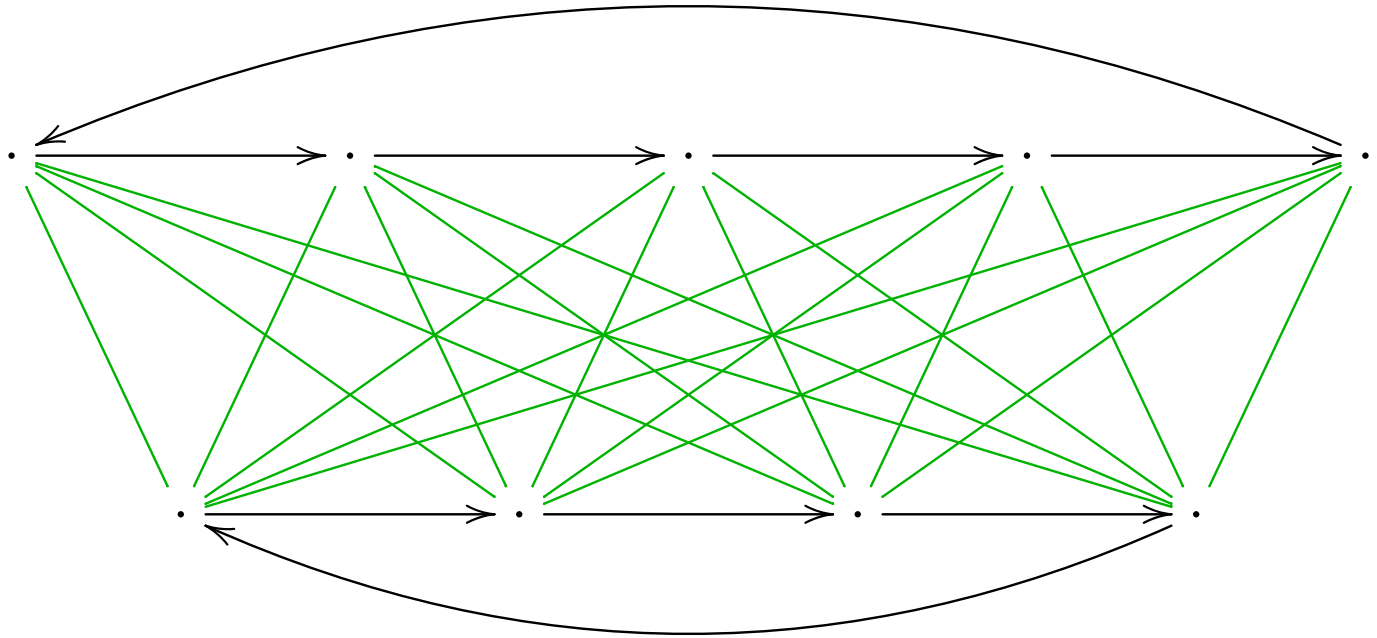
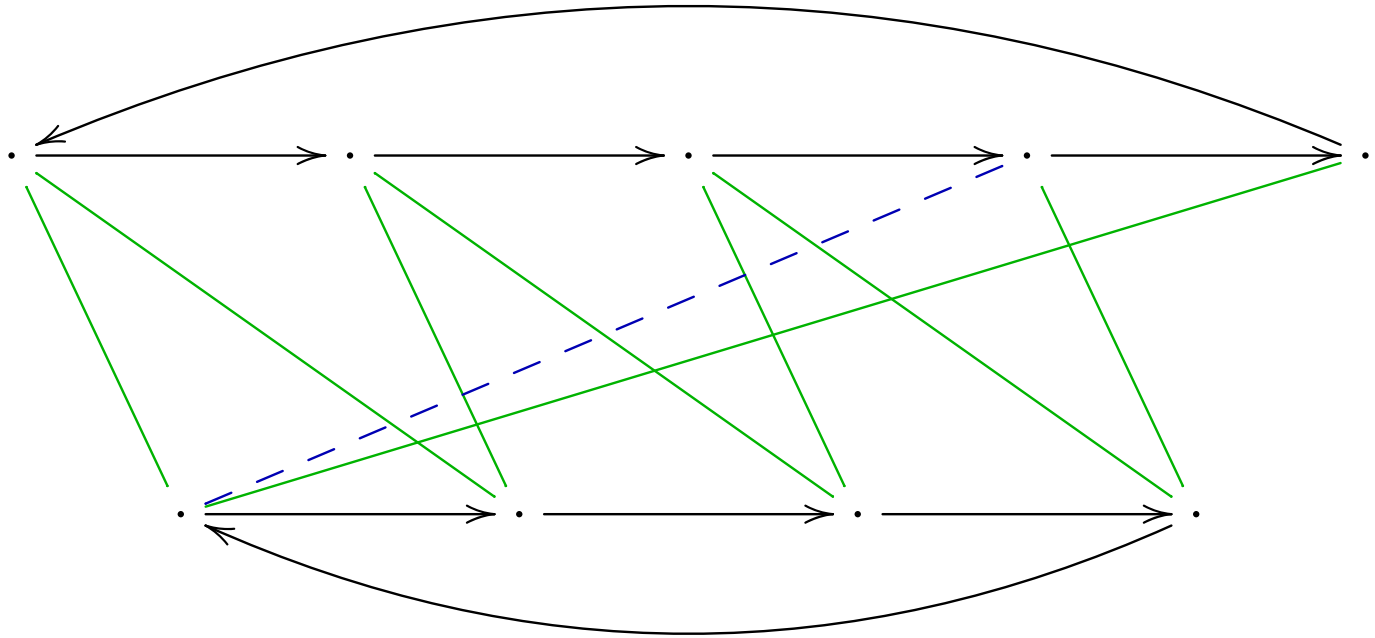


20 pairs

Checking language equivalence

One can stop much earlier



~~20~~ 8 pairs

Checking language equivalence

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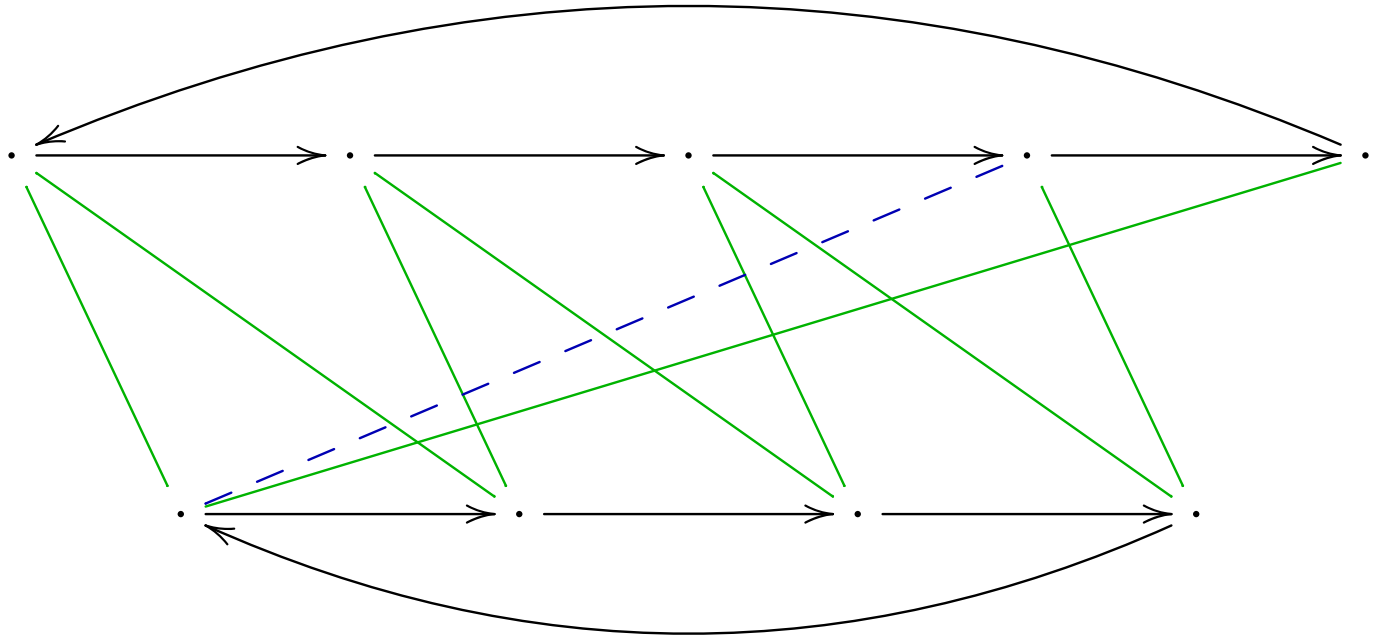


[Hopcroft and Karp '71]
[Tarjan '75]

Complexity: almost linear

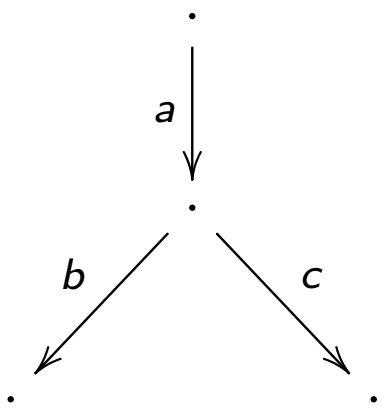
Checking language equivalence

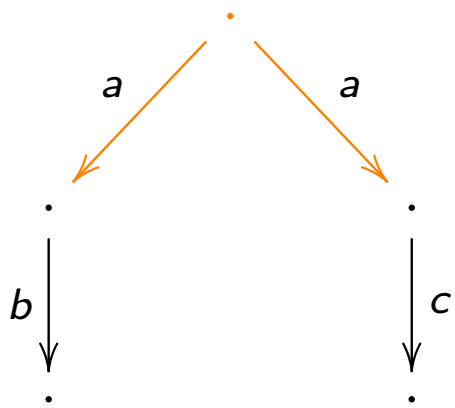
Correctness of HK algorithm, revisited:

bisimulation up to equivalence

Lemma: if R is a bisimulation up to equivalence,
then R^e is a bisimulation

Non-Deterministic Automata





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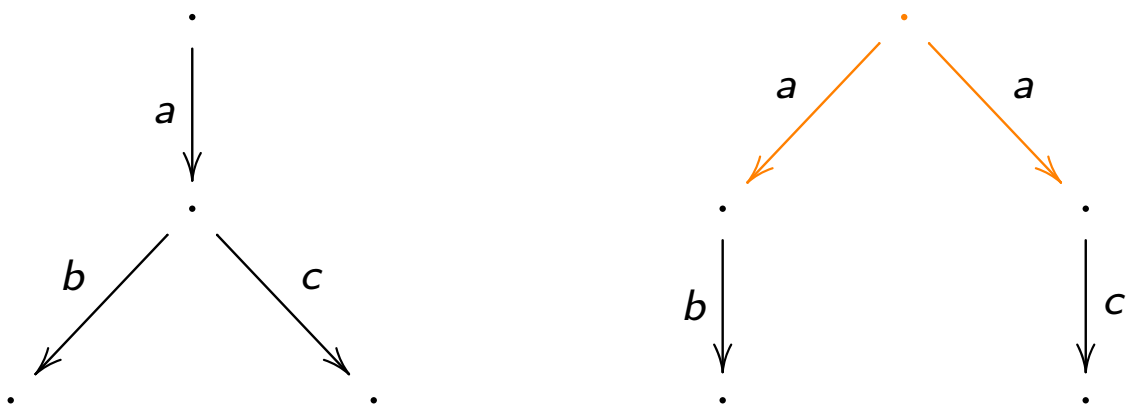
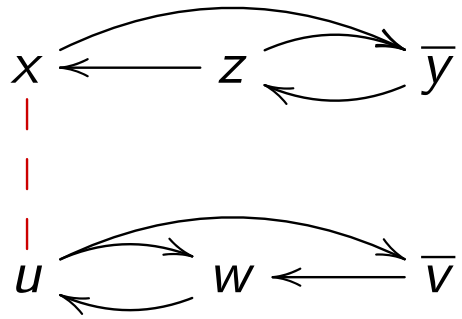


Reduction to the deterministic case:

- "powerset construction": $(S, t, o) \mapsto (\mathcal{P}(S), t^{\#}, o^{\#})$
- from states $(x, y, \dots)$ to sets of states $(X, Y, \dots)$

Checking language equivalence

Non-deterministic case: use Hopcroft and Karp on the fly:



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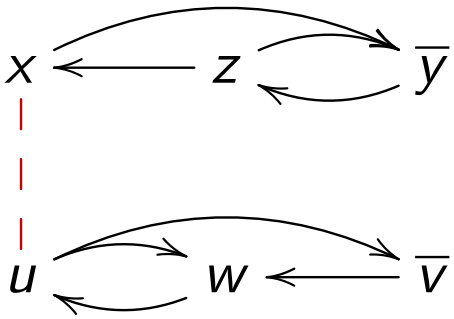

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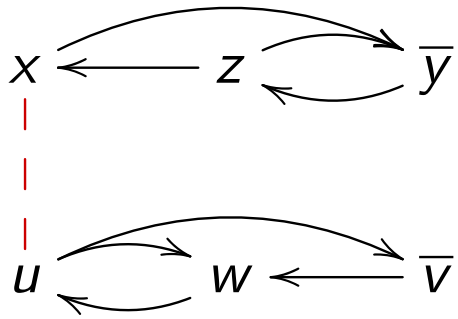

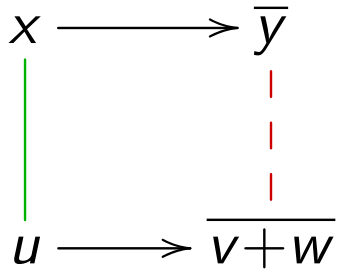

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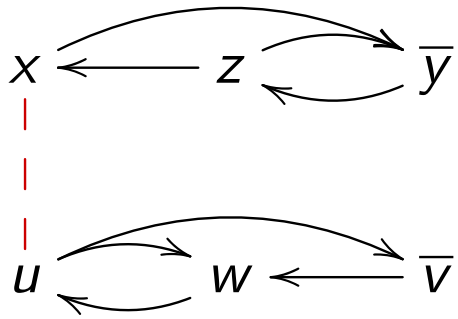

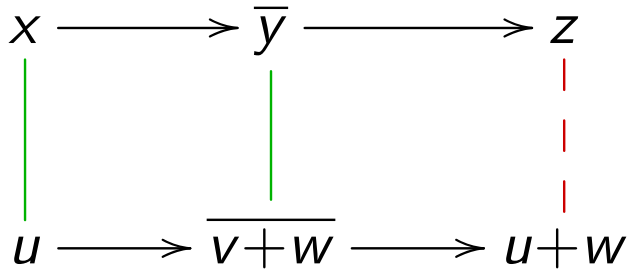

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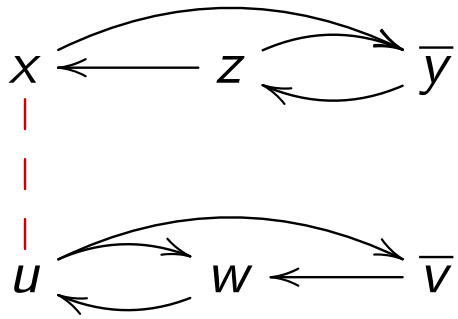

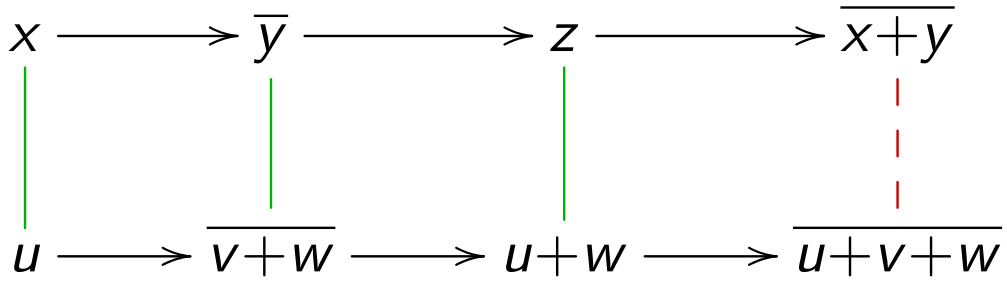

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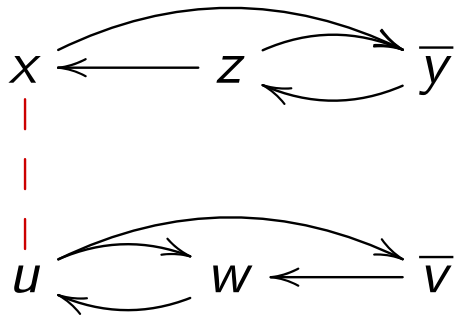

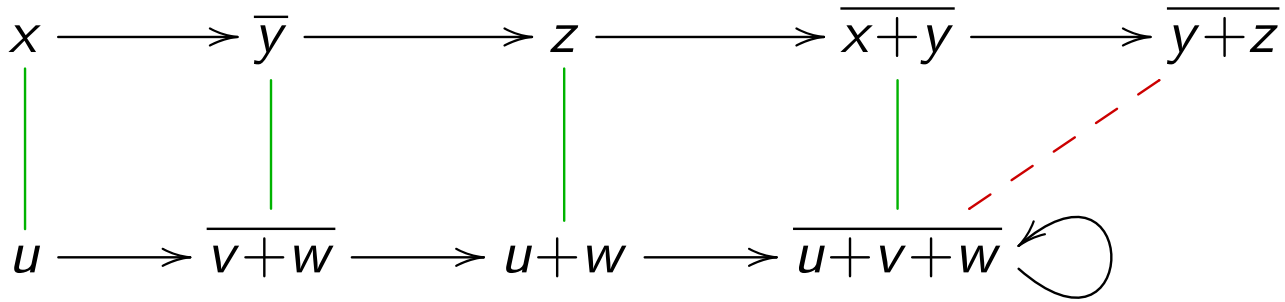

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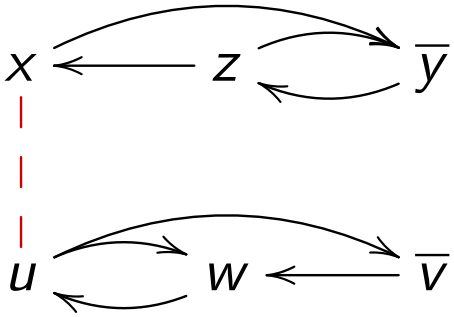

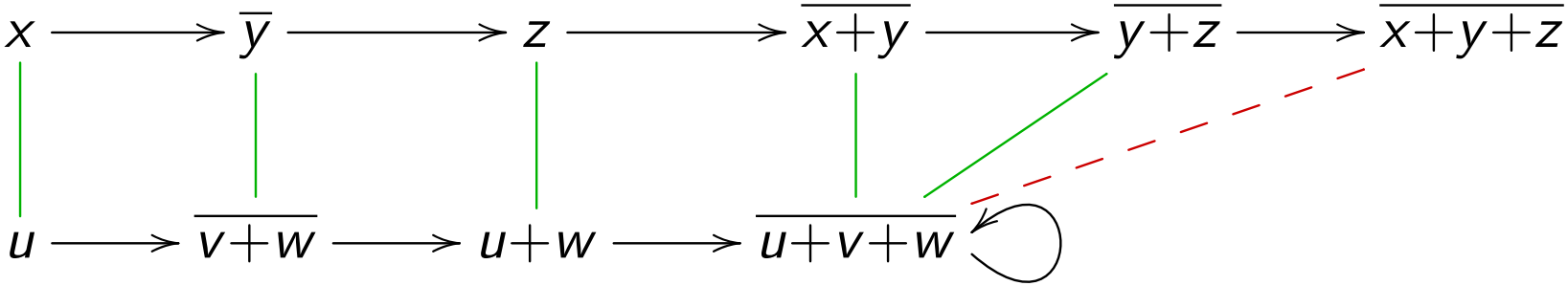

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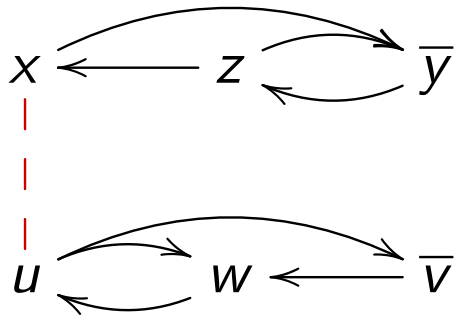

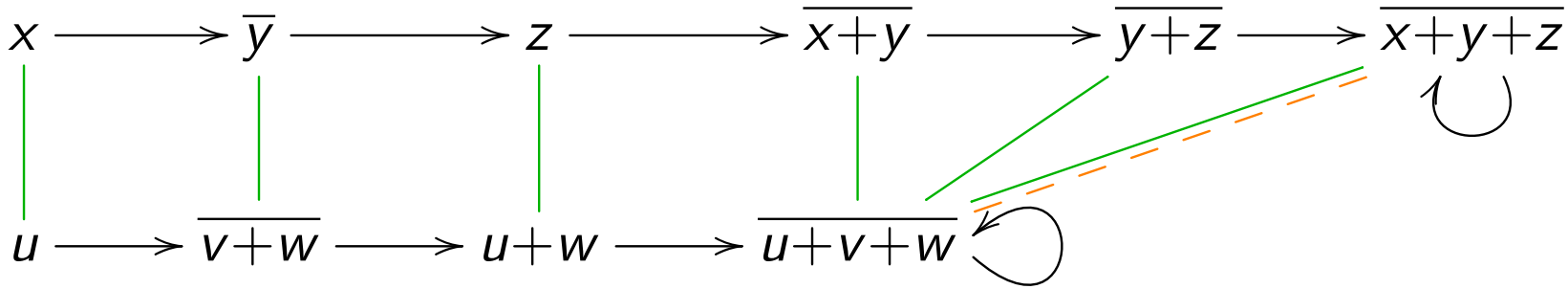

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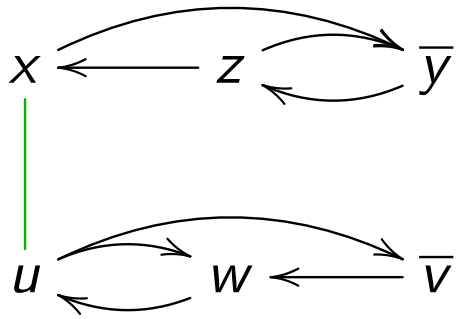

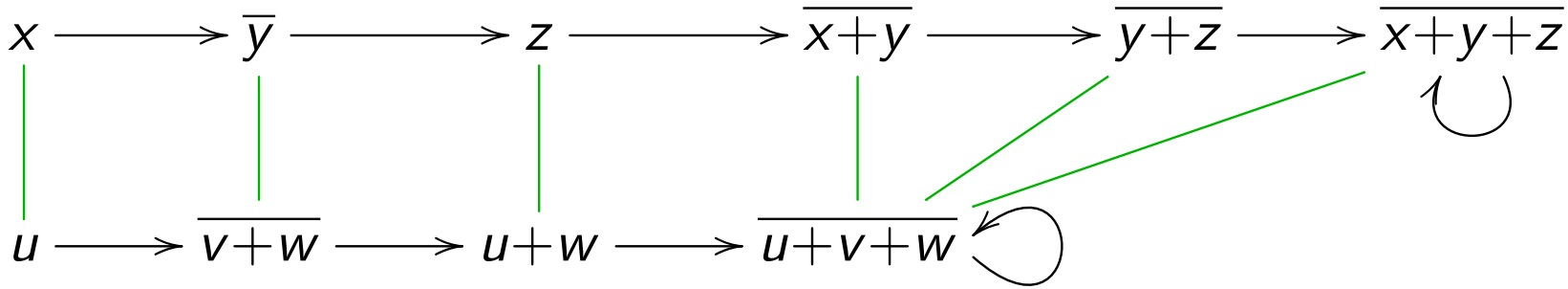

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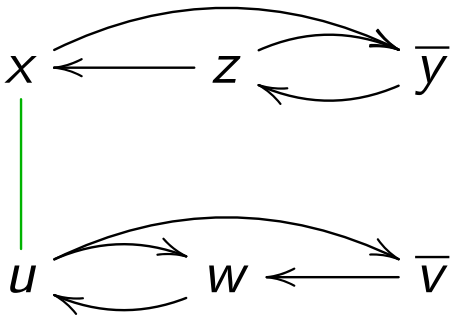
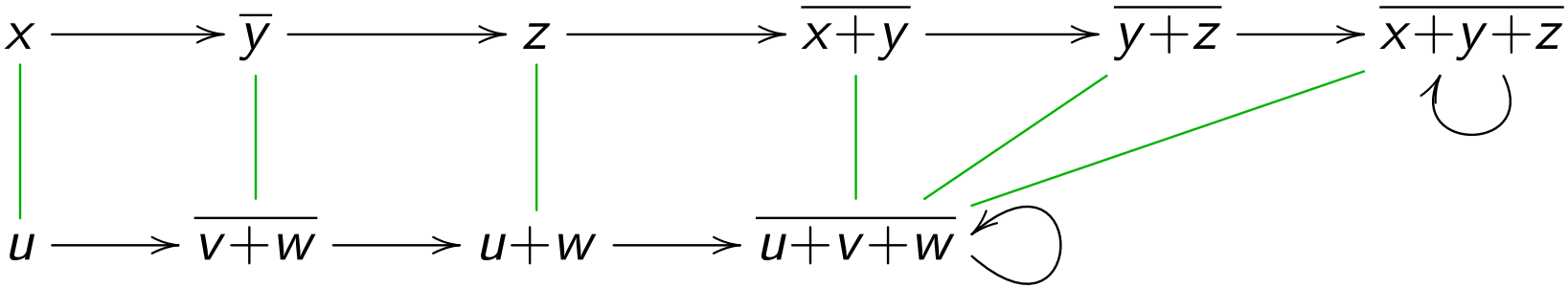
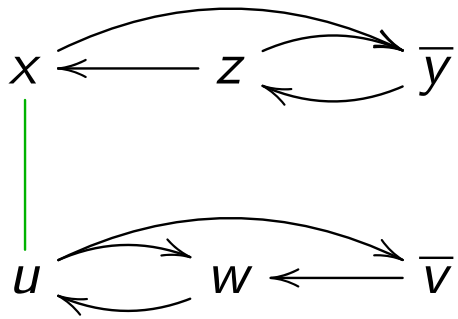


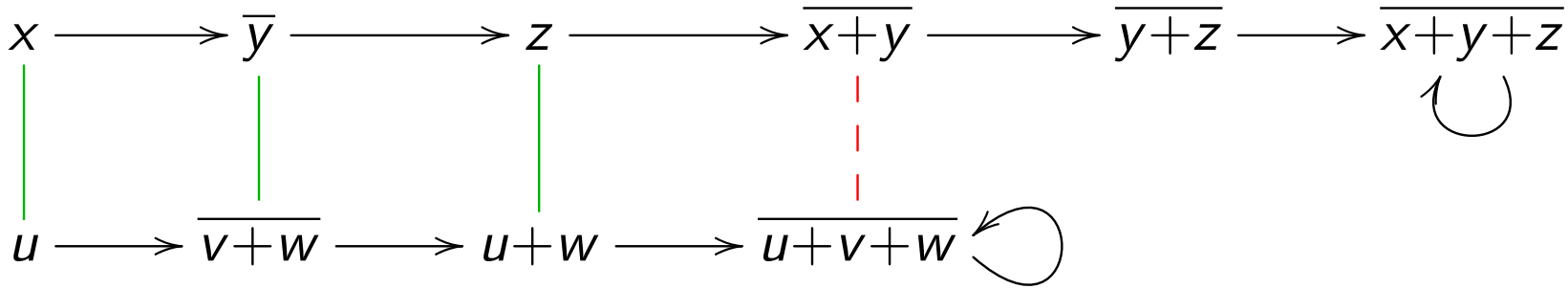


(correctness comes for free)

Checking language equivalence

One can do better:





Checking language equivalence

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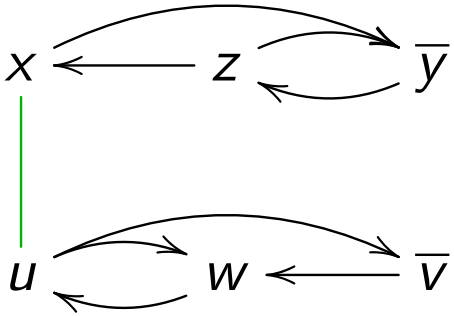
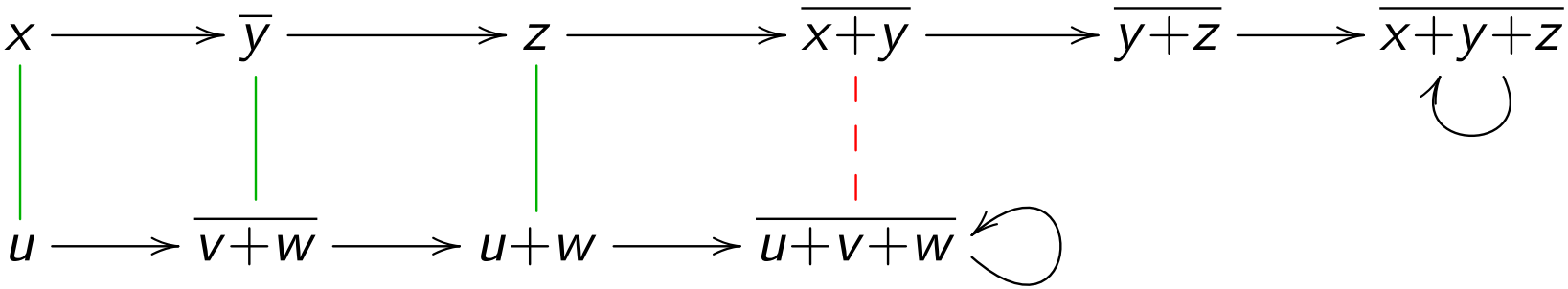


$$\begin{array}{rl} & (x,\ u) \\ + & (y,\ v+w) \\ \hline = & (x+y,\ u+v+w) \end{array}$$



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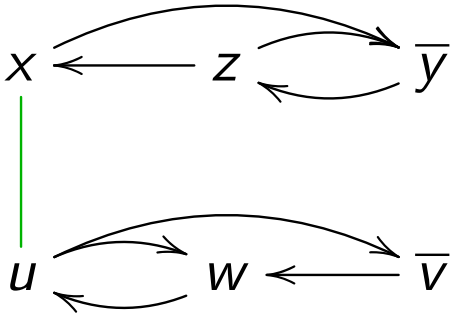

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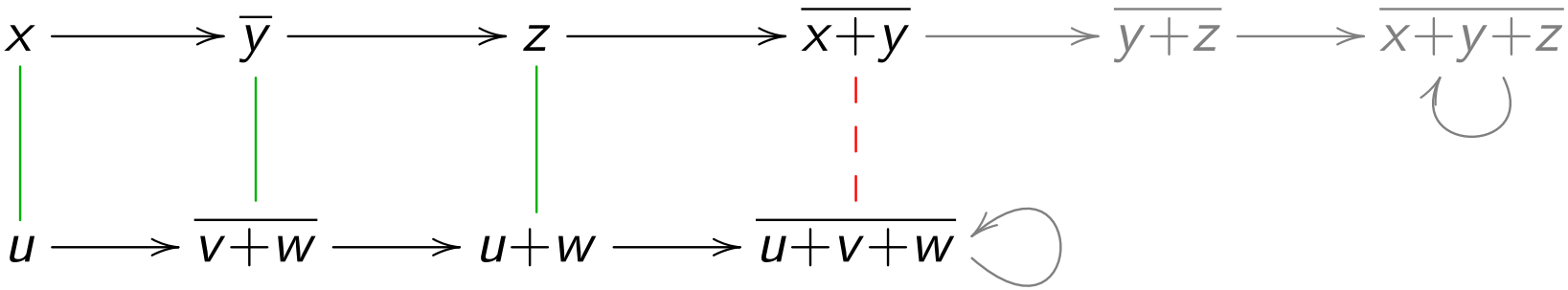


parts of the accessible subsets need not be explored

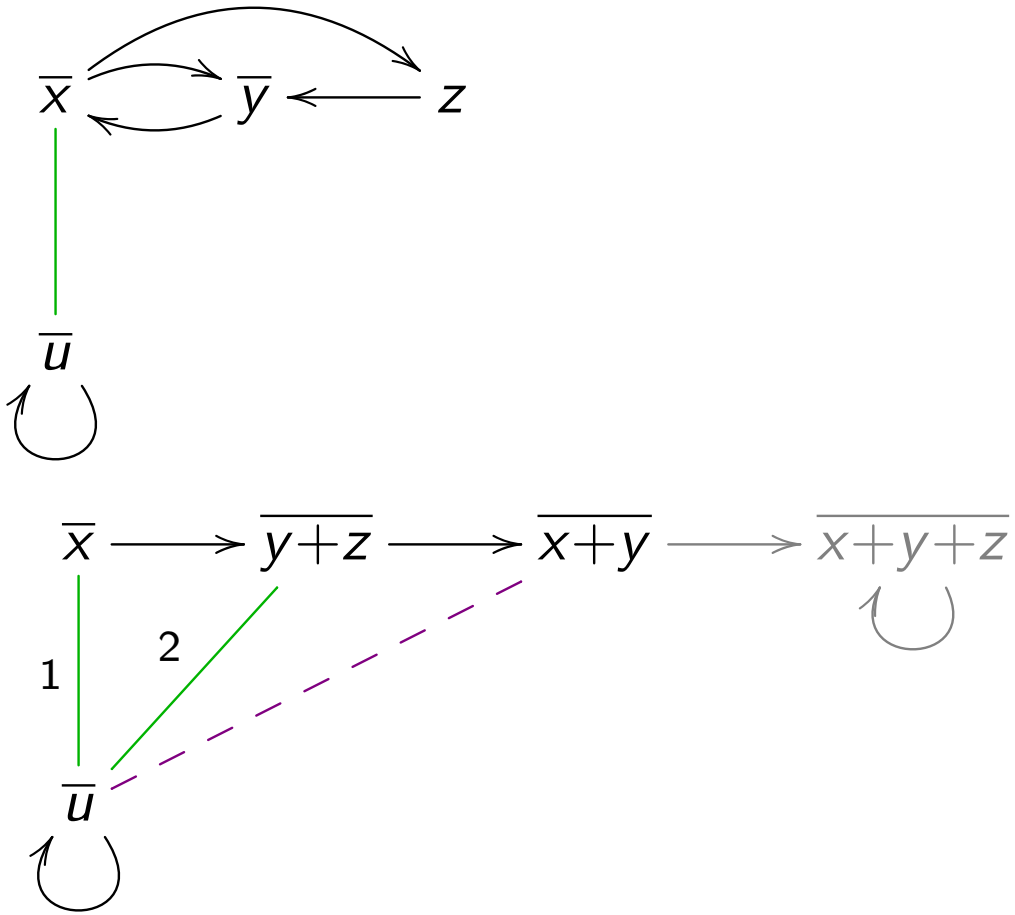
Correctness

bisimulation up to context

Lemma: if R is a bisimulation up to context,
then R^u is a bisimulation

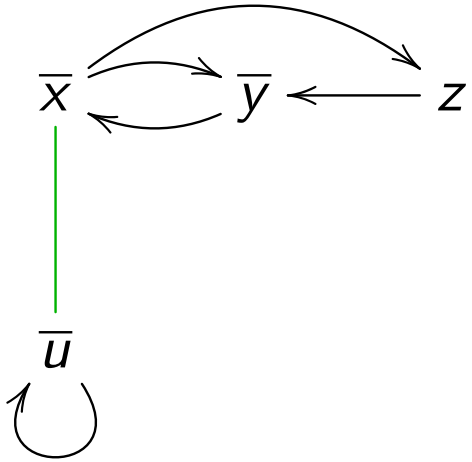
Checking language equivalence

One can do even better:

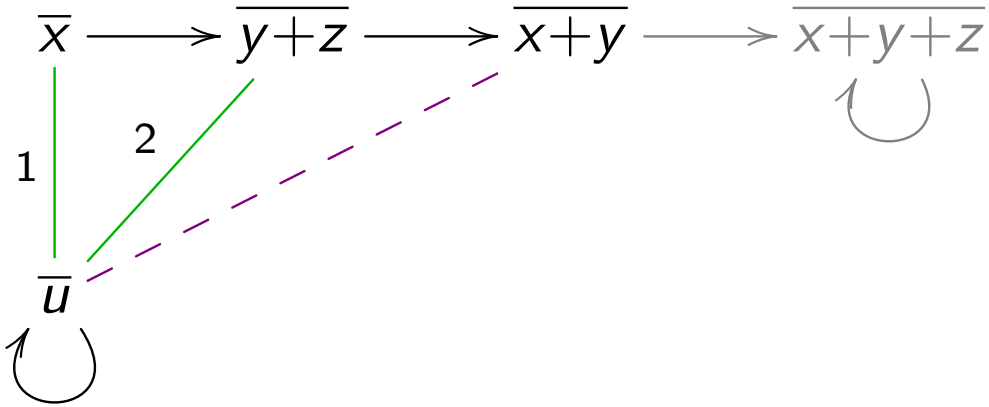


Checking language equivalence

One can do even better:



$$
\begin{aligned}
x+y &= u+y && (1)\\
&= y+z+y && (2)\\
&= y+z\\
&= u && (2)
\end{aligned}
$$



Correctness

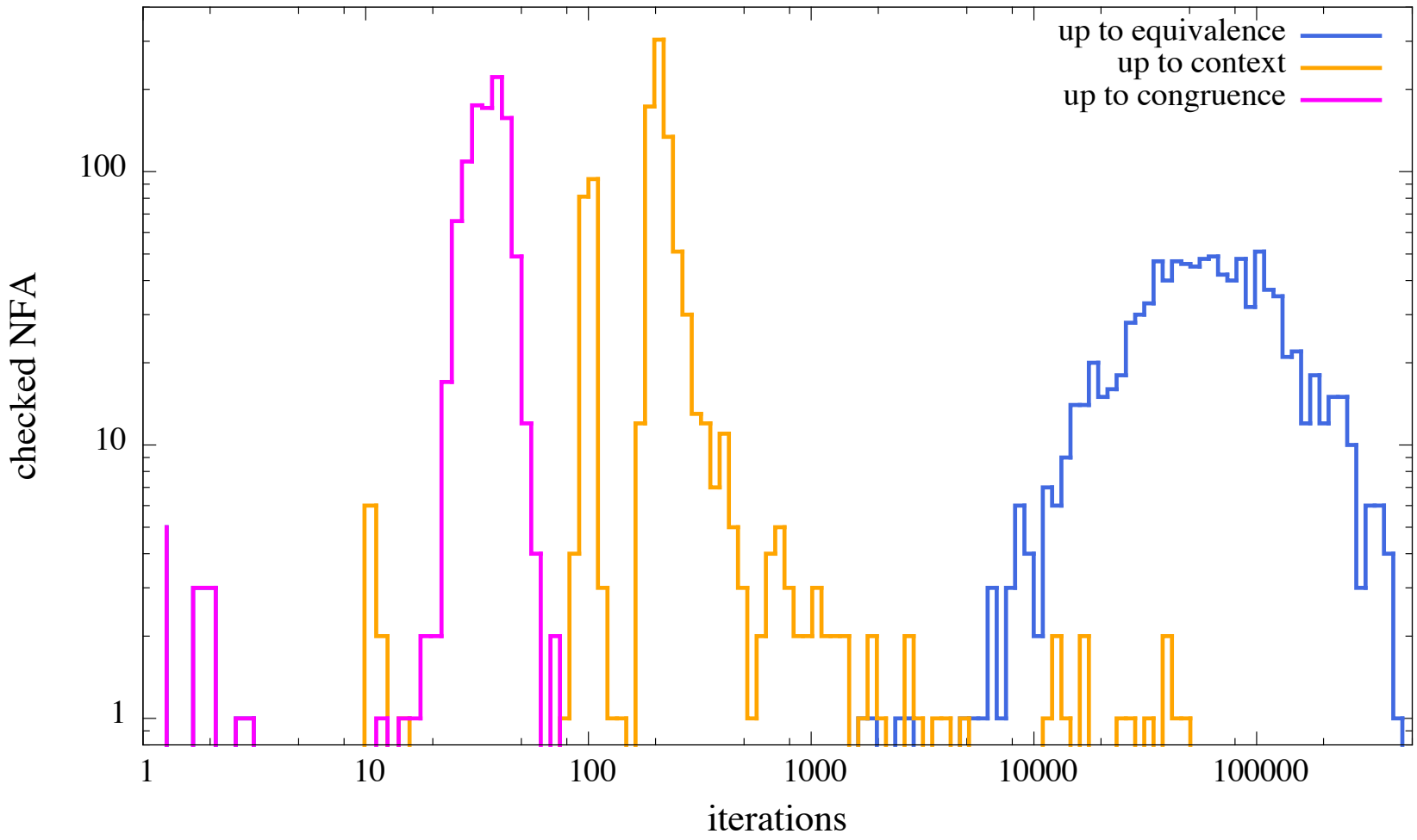
bisimulation up to congruence

Lemma: ...

In practice

http://perso.ens-lyon.fr/damien.pous/hkc/

In practice

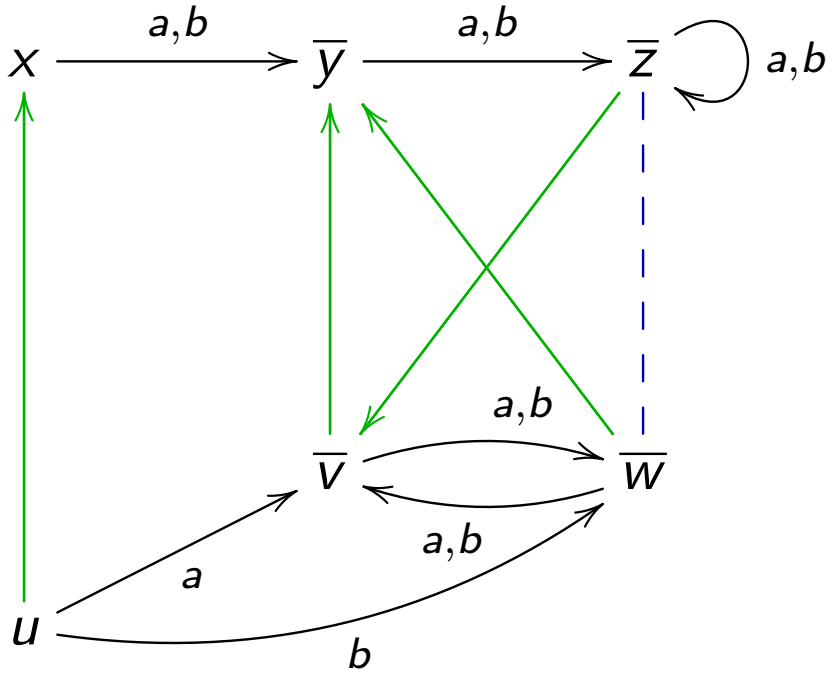

up to equivalence
up to context
up to congruence
checked NFA
100
10
1
1
10
100
1000
10000
100000
iterations

Outline

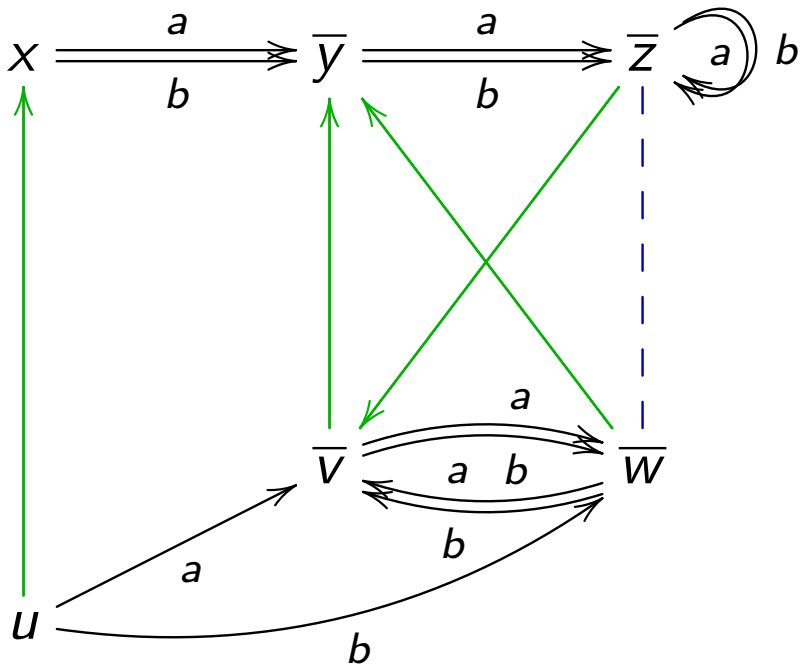
Many states

Many letters

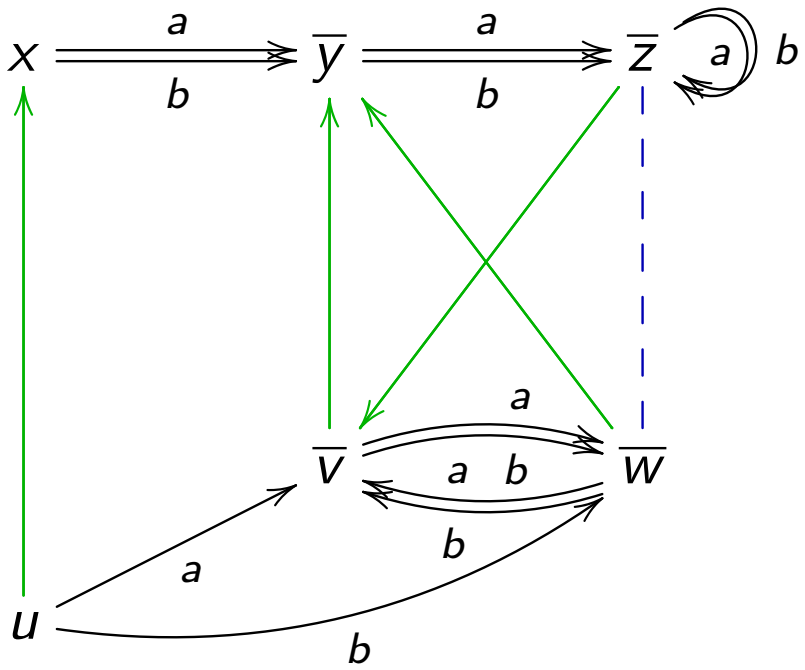
Few states, many letters



Few states, many letters

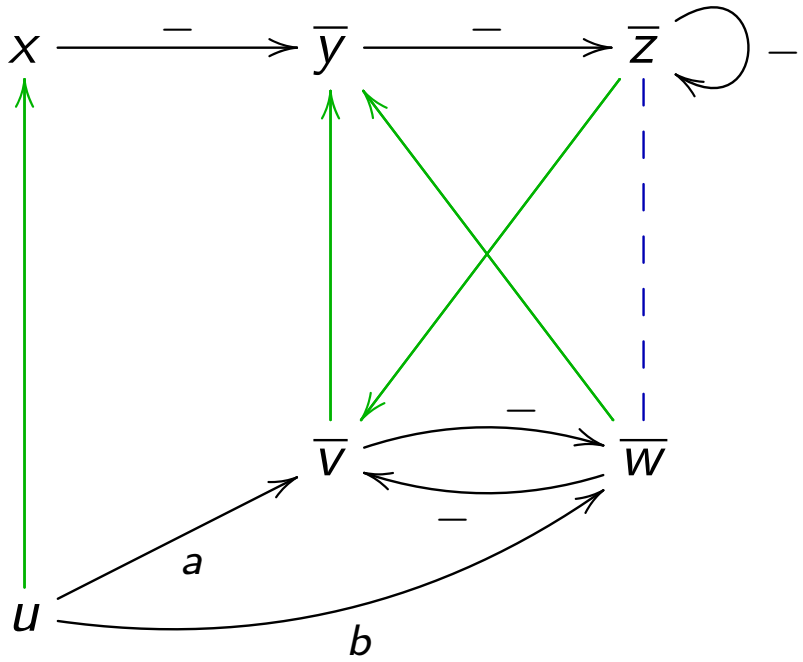


Few states, many letters



- standard practice: label transitions with formulas

Few states, many letters



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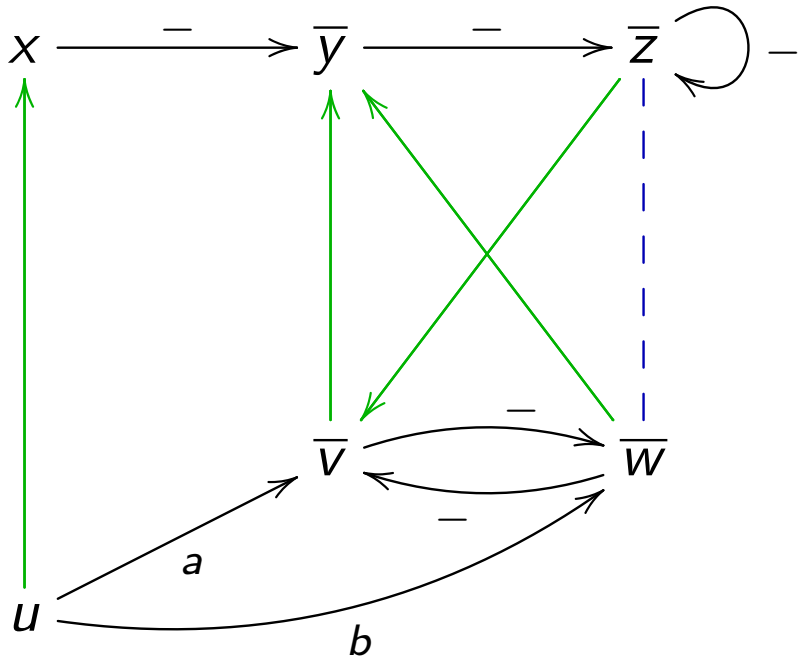
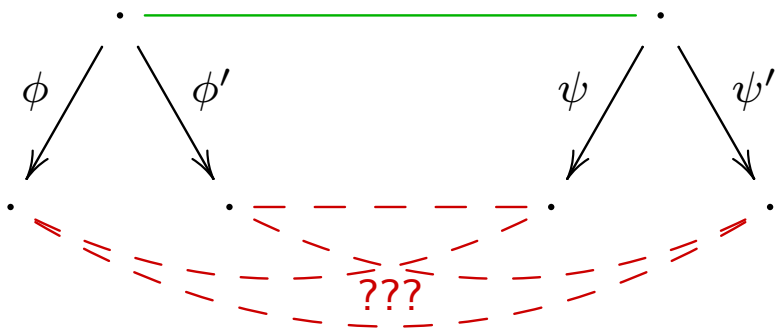


- standard practice: label transitions with formulas
  but then Hopcroft&Karp's algorithm can no longer be used

Few states, many letters



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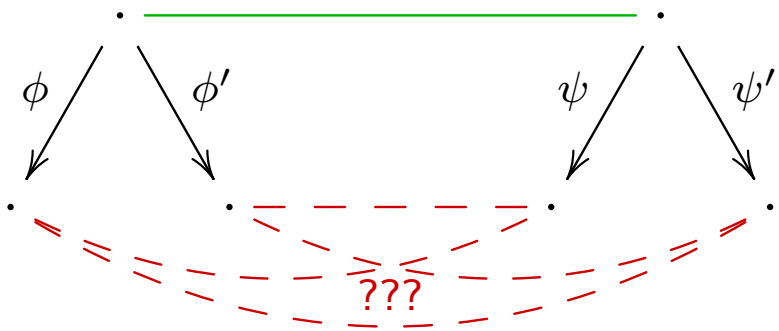
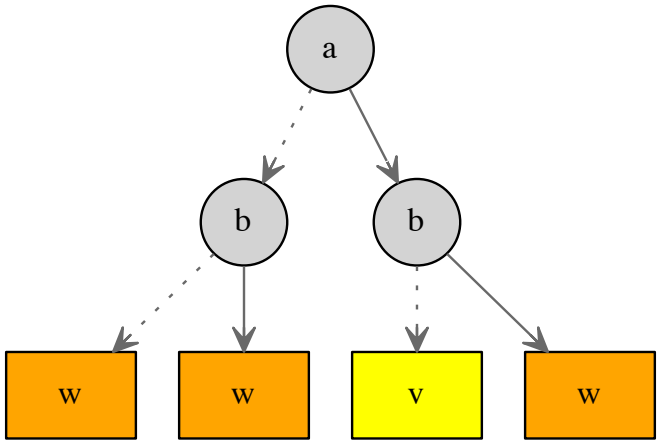


- standard practice: label transitions with formulas
  but then Hopcroft&Karp's algorithm can no longer be used
- more restrictive model here

Binary Decision Diagrams (BDDs) [Bryant'86]

Represent functions of type $2^A \to V$ with compressed decision trees

Example: A={a,b}, V={v,w}, $\alpha \mapsto \begin{cases} v & \text{si } \alpha(a) = 1 \text{ et } \alpha(b) = 0 \\ w & \text{sinon} \end{cases}$



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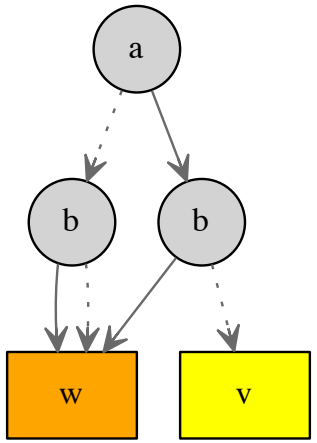

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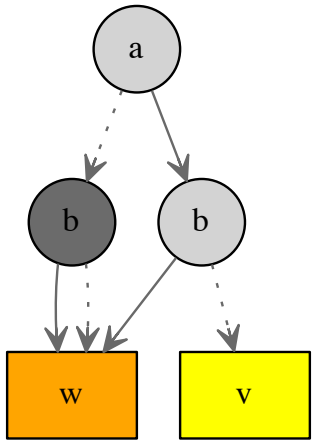

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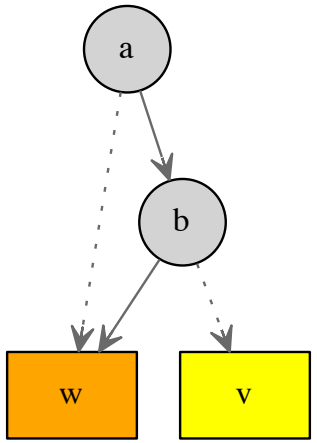


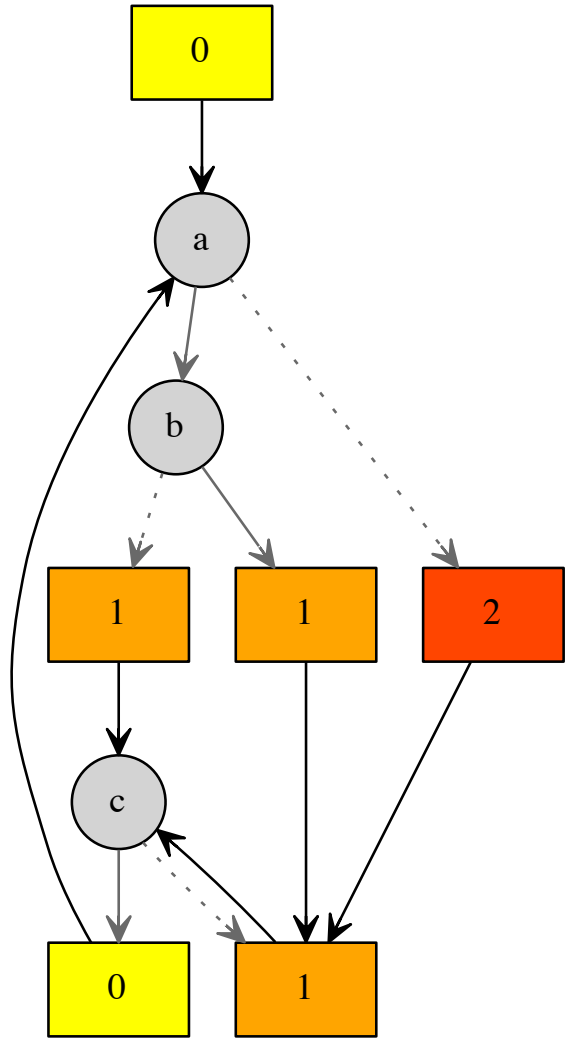
Symbolic Automata

Just represent the transition functions with BDDs

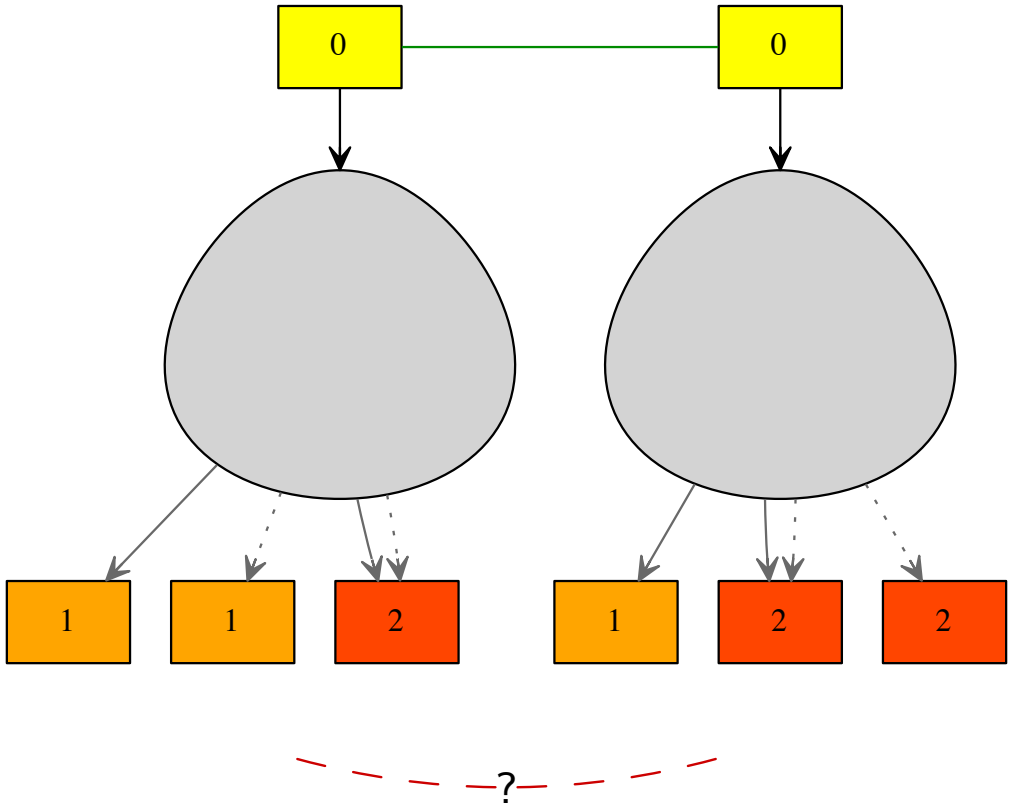
Example:

alphabet: 2^A with $A = \{a, b, c\}$

output set: $\mathbb{N}$



Iterate over the successors of two states



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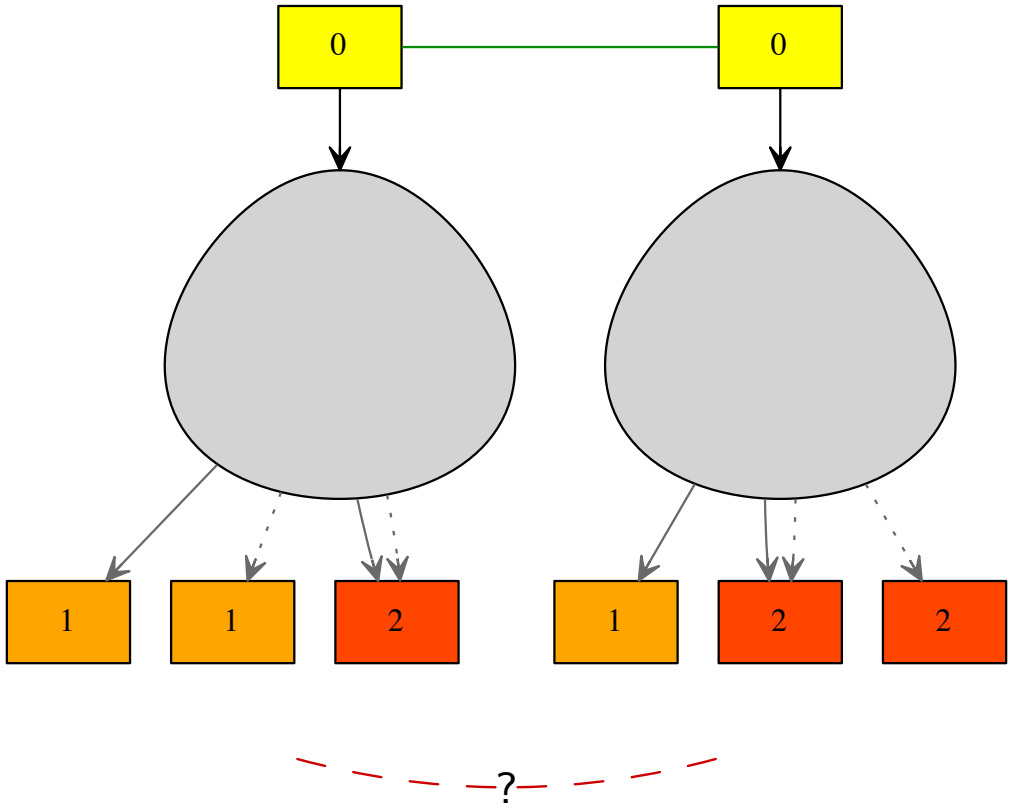
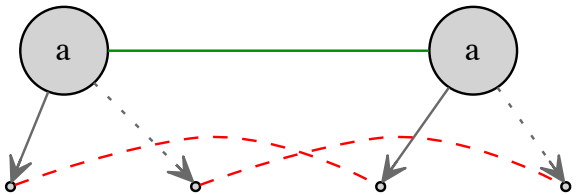
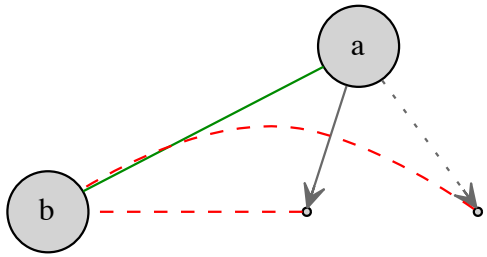
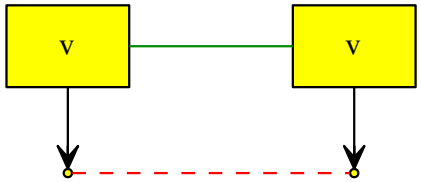


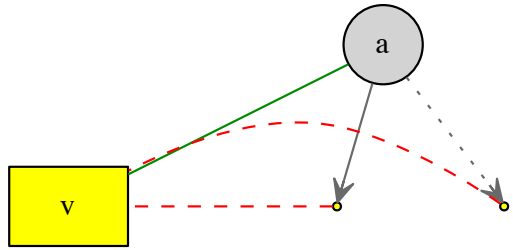
We just need an "`iter2`" function on BDDs

iter2 on BDDs

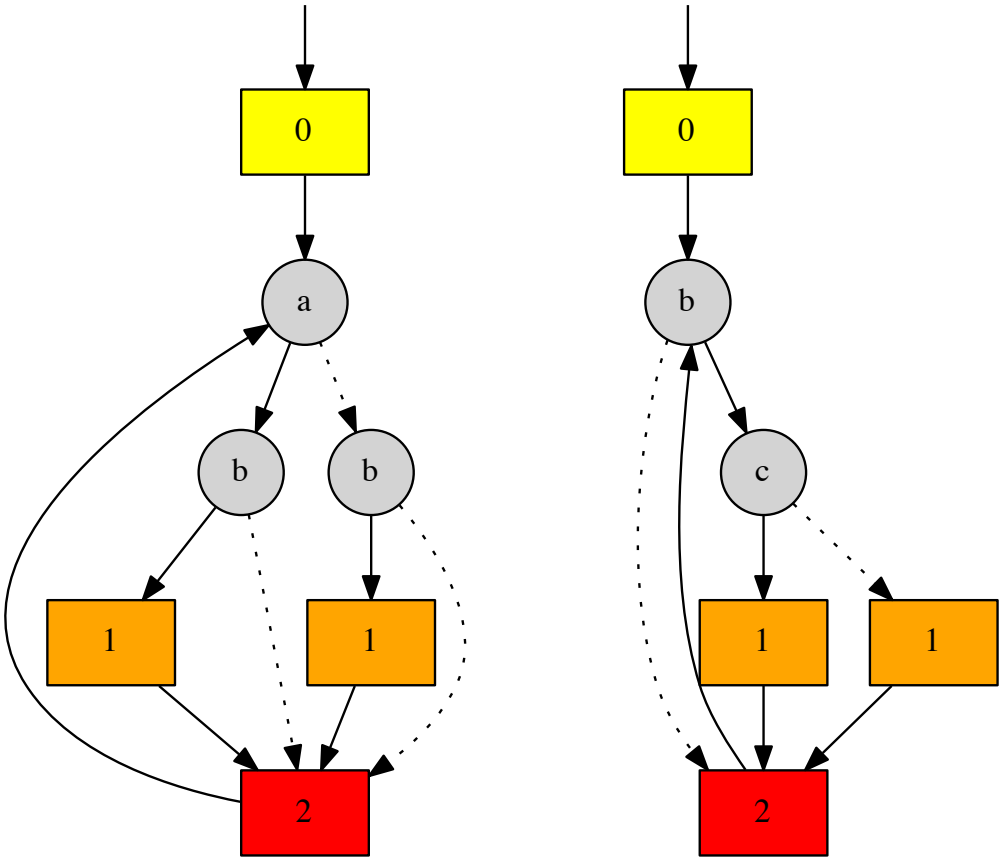








Symbolic algorithm



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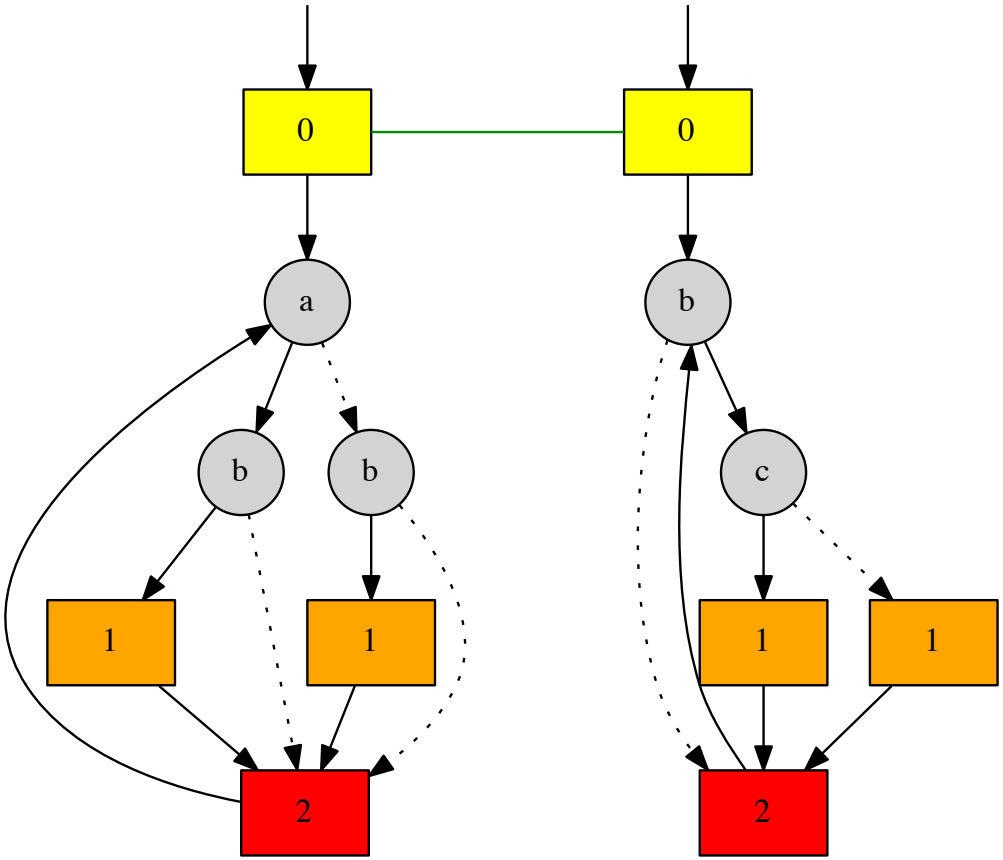
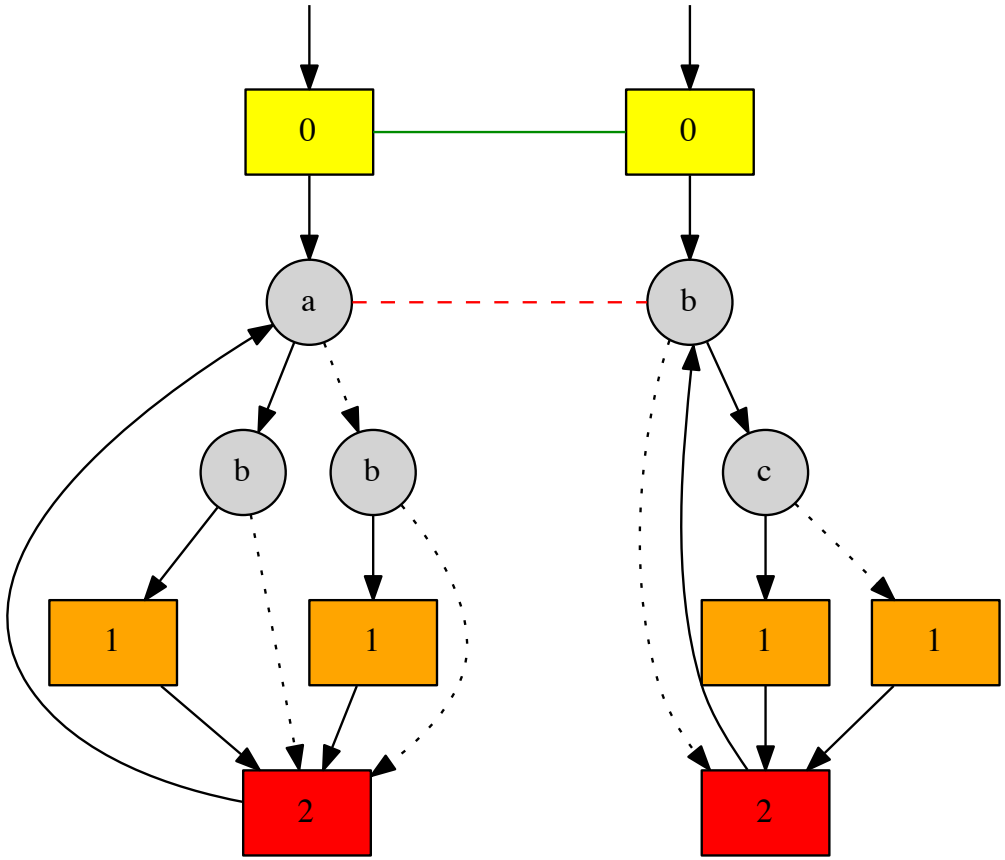
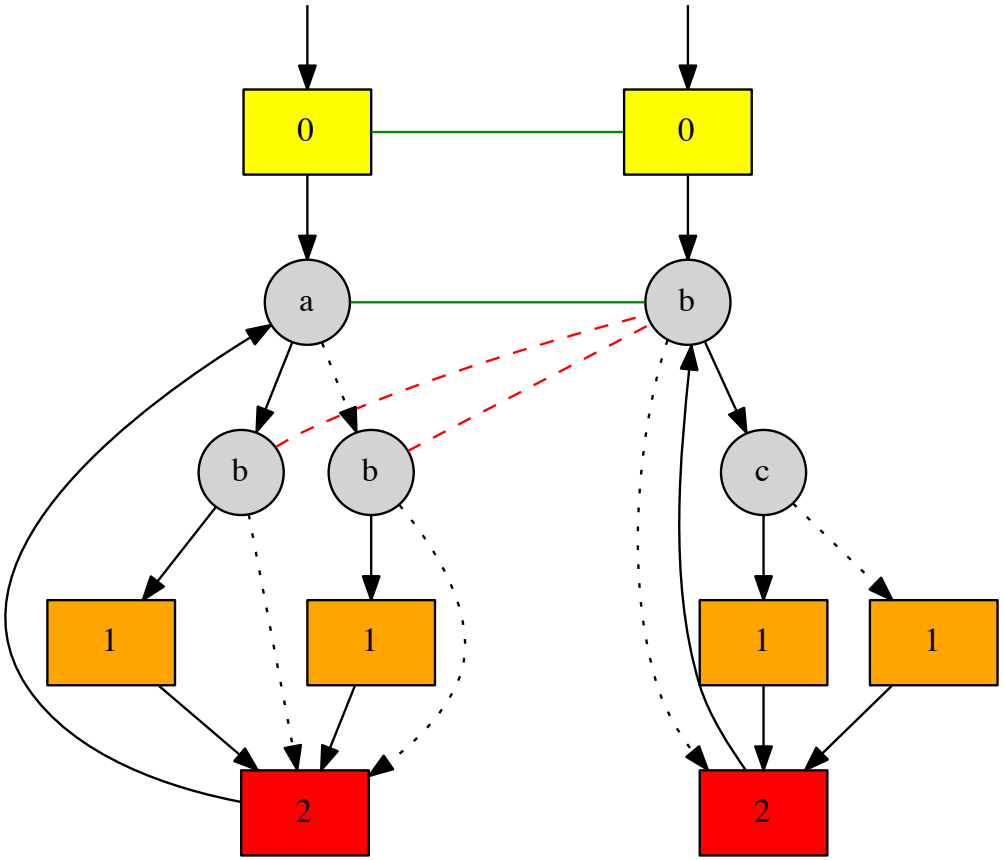


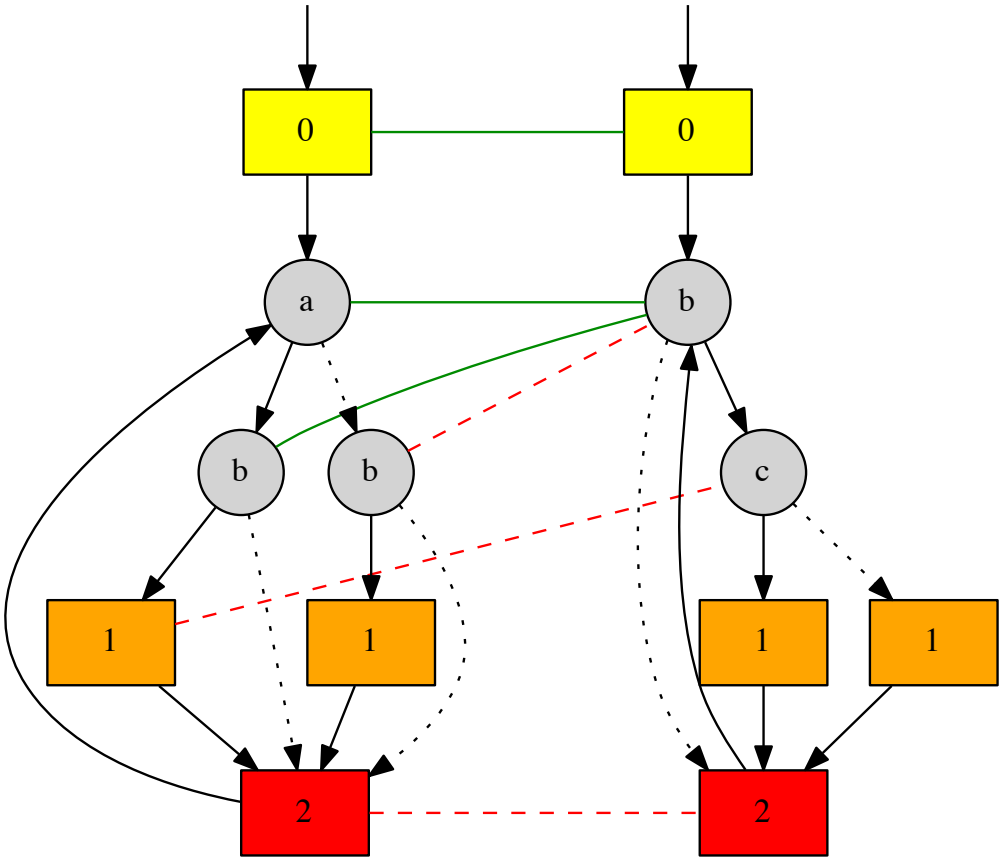
Symbolic algorithm



Symbolic algorithm



Symbolic algorithm



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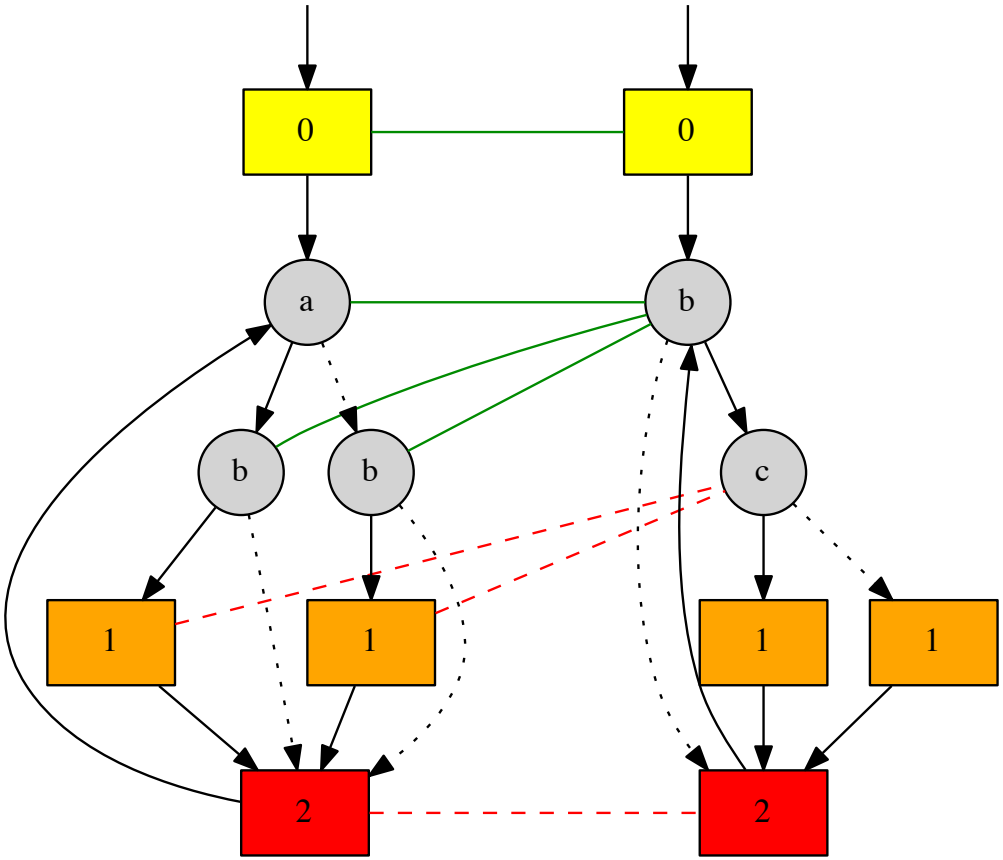

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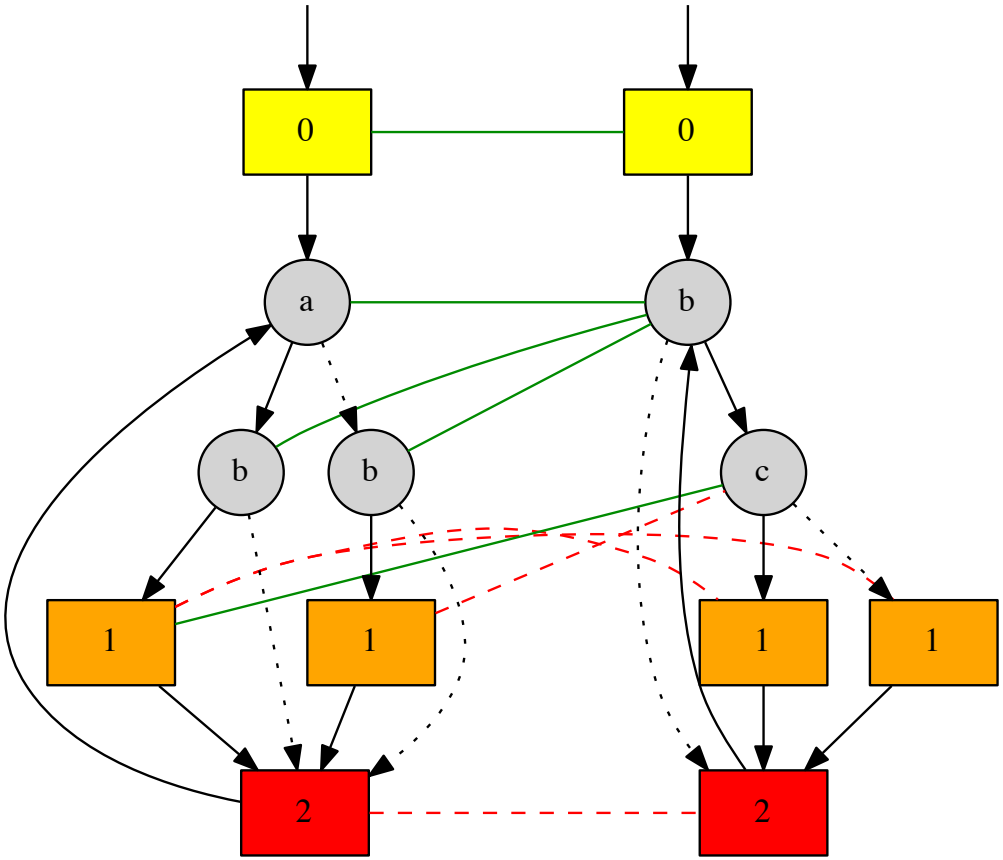
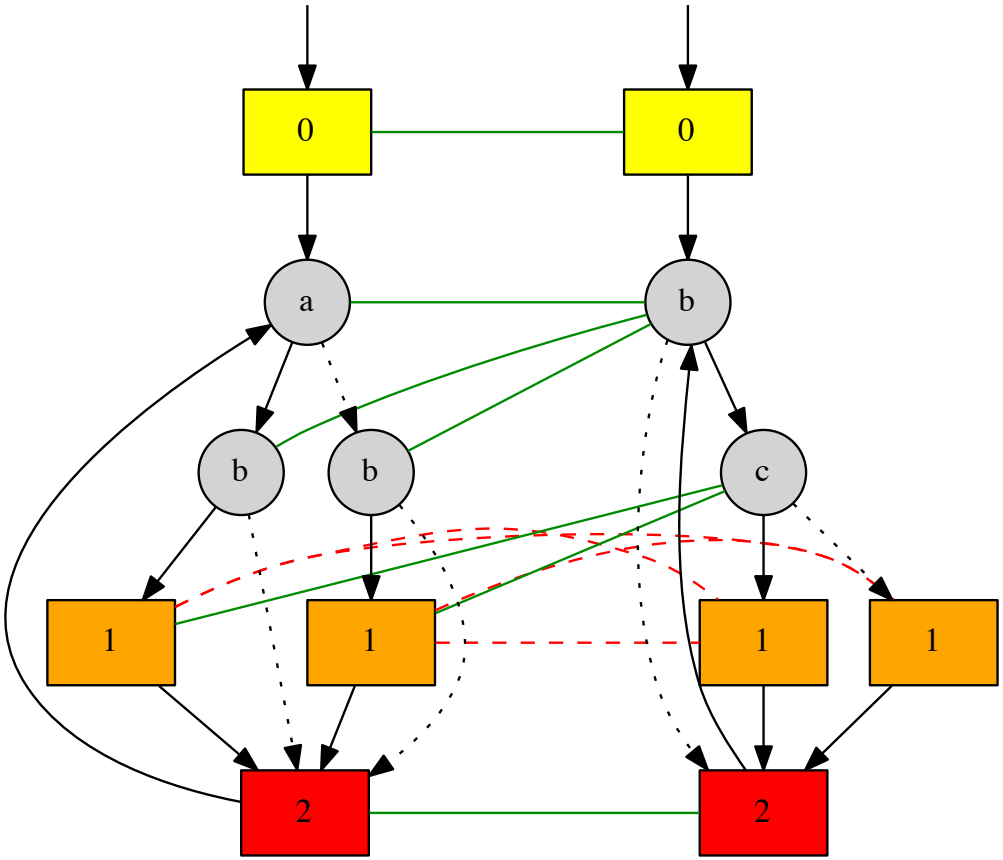
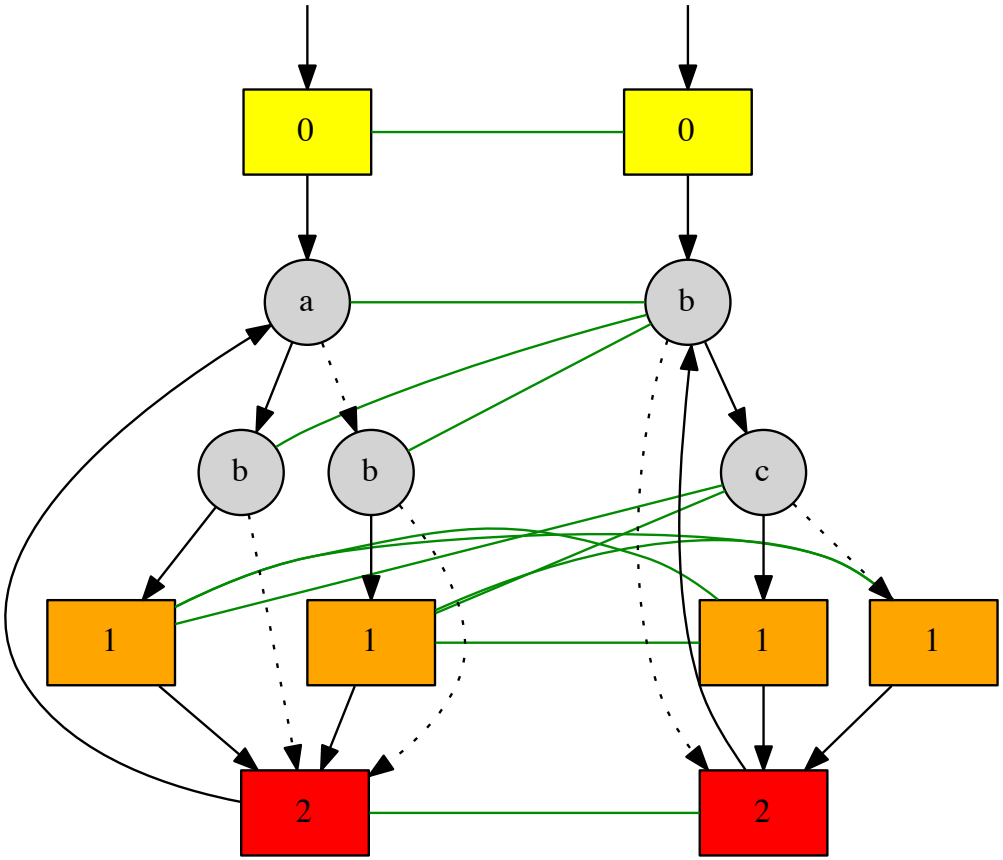


Symbolic algorithm



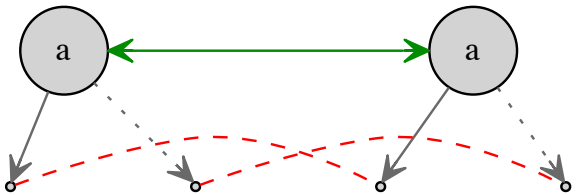
Symbolic algorithm

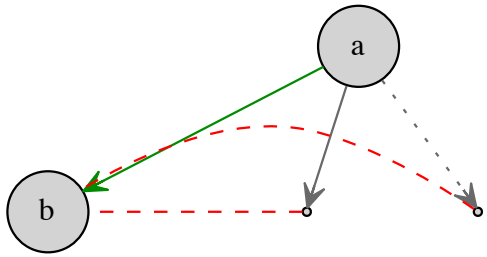


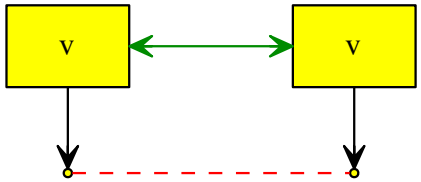
How to incorporate up to techniques?

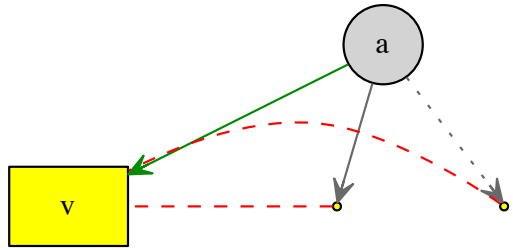
unifying BDDs

The union-find datastructure can be used between BDD nodes!

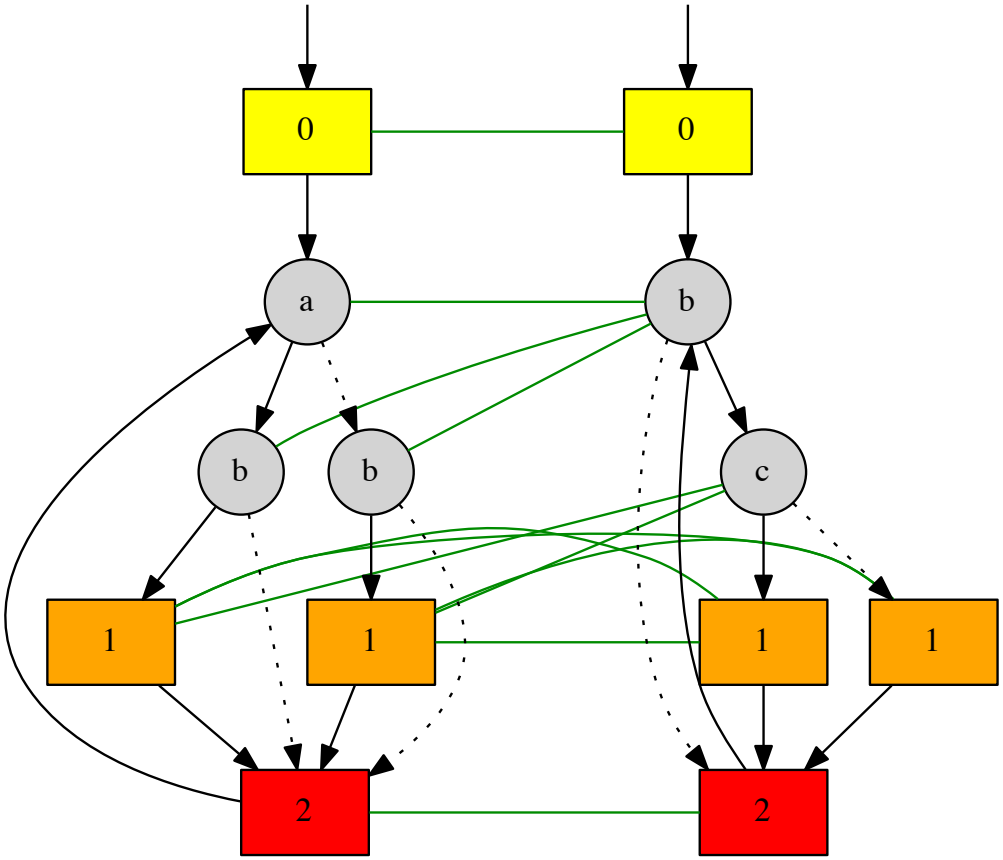




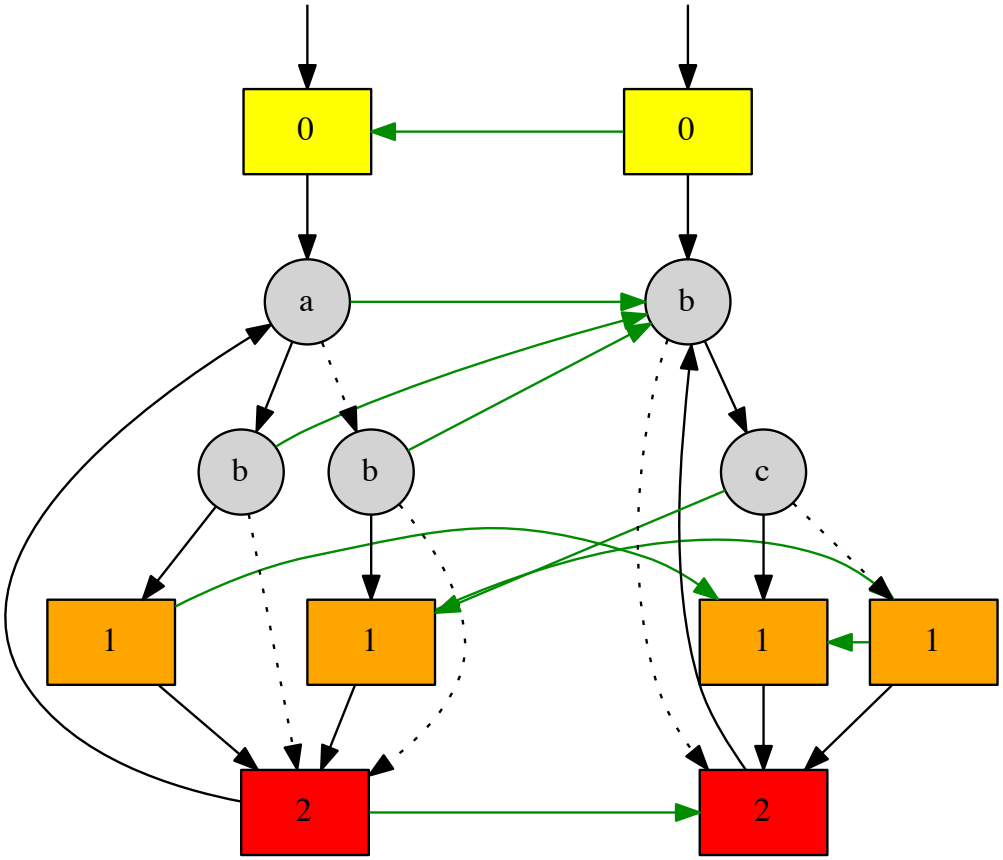




Symbolic Hopcroft & Karp's algorithm



Symbolic Hopcroft & Karp's algorithm



In practice

Applied to Kleene algebra with tests (KAT)

`http://perso.ens-lyon.fr/damien.pous/symkat/`

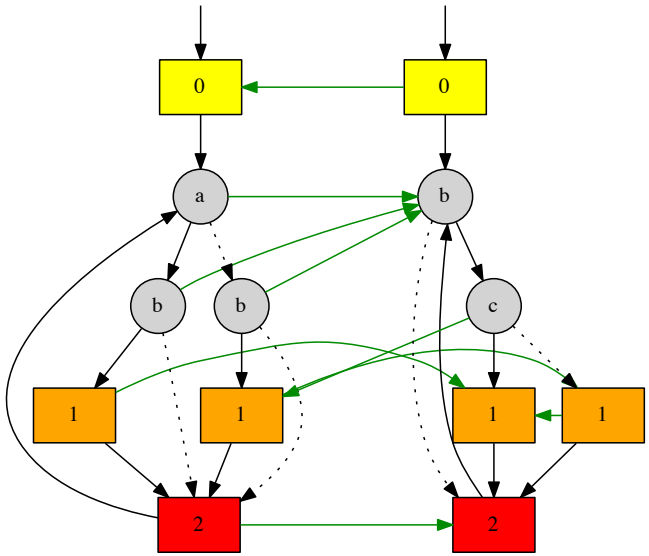
Complexity?

- Average complexity?
- Constrained union-find?

To remember

- Improvement of Hopcroft & Karp's algorithm [Bonchi, Pous '13]
  $\rightarrow$ using bisimulations up to congruence

- Symbolic version of Hopcroft & Karp's algorithm [Pous '15]
  $\rightarrow$ mixing BDDs and Union-Find



`http://perso.ens-lyon.fr/damien.pous/hkc/`
`http://perso.ens-lyon.fr/damien.pous/symkat/`