Recurrence function of Sturmian sequences. A probabilistic study

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Ongoing work with
Valérie Berthé, Eda Cesaratto, Brigitte Vallée, and Alfredo Viola

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Study in combinatorics of words.

Main aim: description of the finite factors of an infinite word u

- How many factors of length n? $\longrightarrow$ Complexity
- What are the gaps between them? $\longrightarrow$ Recurrence

Very easy when the word is eventually periodic !

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what are the extreme bounds for the recurrence function?

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Here, in a convenient model,

we perform a probabilistic study:

For a "random" sturmian word, and for a given "position",

– what is the mean value of the recurrence?

– what is the limit distribution of the recurrence?

Plan of the talk

Complexity

$\mathcal{L}_u(n)$ denotes the set of factors of length n in u.

Definition

Complexity function of an infinite word $u \in \mathcal{A}^{\mathbb{N}}$

$$p_u : \mathbb{N} \to \mathbb{N}\,, \qquad p_u(n) = |\mathcal{L}_u(n)|\ .$$

Two simple facts: $\quad p_u(n) \leq |\mathcal{A}|^n\,, \qquad p_u(n) \leq p_u(n+1)\,.$

Important property

$u \in \mathcal{A}^{\mathbb{N}}$ is not eventually periodic

$\iff p_u(n+1) > p_u(n)$

$\implies p_u(n) \geq n+1\,.$

Recurrence

Definition (Uniform recurrence)

A word $u \in \mathcal{A}^{\mathbb{N}}$ is uniformly recurrent iff each finite factor appears infinitely often and with bounded gaps.

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Let $u \in \mathcal{A}^{\mathbb{N}}$ be uniformy recurrent. The recurrence function of u is:

$$R_{\langle u \rangle}(n) = \inf \{ m \in \mathbb{N} : \text{any } w \in \mathcal{L}_u(m) \text{ contains all the factors } v \in \mathcal{L}_u(n) \} .$$

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An important inequality between the two functions,

the complexity function and the recurrence function

$$R_{\langle u \rangle}(n) \geq p_u(n) + n - 1.$$

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Explicit construction

Associate with a pair (α, β) the two sequences

$$\underline{u}_n = \lfloor \alpha\,(n+1) + \beta \rfloor - \lfloor \alpha\, n + \beta \rfloor$$

$$\overline{u}_n = \lceil \alpha\,(n+1) + \beta \rceil - \lceil \alpha\, n + \beta \rceil$$

and the two words $\underline{S}(\alpha, \beta)$ and $\overline{S}(\alpha, \beta)$ produced in this way.

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A word u is Sturmian iff there are $\alpha, \beta \in [0, 1[$, with α irrational, such that $u = \underline{S}(\alpha, \beta)$ or $u = \overline{S}(\alpha, \beta)$.

Recurrence of Sturmian words

Property

Let u be a Sturmian word of the form $\underline{S}(\alpha, \beta)$ or $\overline{S}(\alpha, \beta)$. Then

- u is uniformly recurrent
- $R_{\langle u \rangle}(n)$ only depends on α, and it is written as $R_{\alpha}(n)$.

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Reminder :
The continuant $q_k(\alpha)$ is the denominator of the k-th convergent of α.
It is obtained via the truncation at depth k of the CFE of α.

The sequence $(q_k(\alpha))_k$ is strictly increasing.

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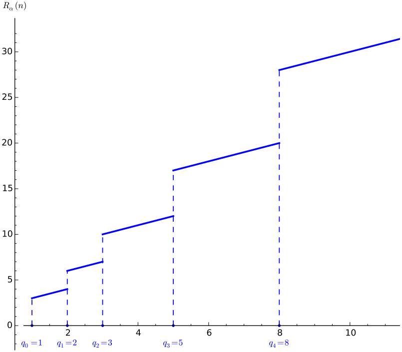
Theorem (Morse, Hedlund, 1940)

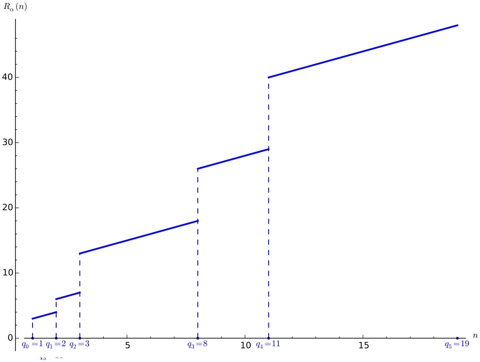
The recurrence function is piecewise affine and satisfies

$$R_{\alpha}(n) = n - 1 + q_{k-1}(\alpha) + q_k(\alpha), \qquad \text{for } n \in [q_{k-1}(\alpha), q_k(\alpha)[.$$

Recurrence function for two Sturmian words



Recurrence function for $\alpha = \varphi^2$,
with $\varphi = (\sqrt{5}-1)/2$.



Recurrence function for $\alpha = 1/e$.

Recurrence function of Sturmian words: classical results.

Proposition

For any irrational $\alpha \in [0,1]$*, one has* $$\liminf \frac{R_\alpha(n)}{n} \leq 3\,.$$

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Theorem

For almost any irrational α, *one has*

$$\limsup \frac{R_\alpha(n)}{n\,\log n} = \infty, \qquad \limsup \frac{R_\alpha(n)}{n\,(\log n)^c} = 0 \quad \textit{for any } c > 1$$

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Proof: Apply the Morse–Hedlund formula and Khinchin's Theorem.

Our point of view

Usual studies of $R_\alpha(n)$

- consider all possible sequences of indices n.
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Here:

- we study particular sequences of indices n depending on α, defined with their position on the intervals $[q_{k-1}(\alpha), q_k(\alpha)[$.
- we then draw α at random.
- we perform a probabilistic study.
- we then study the role of the position
  in the probabilistic behaviour of the recurrence function.

Subsequences with a fixed position

We work with particular subsequences of indices n

Given $\mu \in]0, 1]$ the sequence

$$n_k^{\langle \mu \rangle}(\alpha) = q_{k-1}(\alpha) + \lfloor \mu \, (q_k(\alpha) - q_{k-1}(\alpha)) \rfloor$$

is the subsequence of position μ of α.

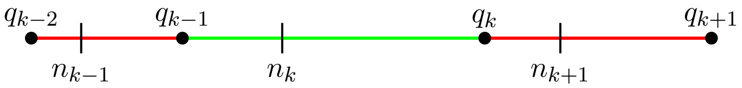

Figure: Sequence of indices n for $\mu = 1/3$.

We study

- the behaviour of

$$\frac{R_\alpha(n)}{n}, \quad n = n_k^{\langle\mu\rangle} = q_{k-1} + \lfloor \mu \left(q_k - q_{k-1} \right) \rfloor$$

  when n has a fixed position μ within $[q_{k-1}, q_k[$.

  Remark that $(n_k^{\langle\mu\rangle})_k$ is a sequence depending on $\alpha \in \mathcal{I}$.

- what happens when α is drawn uniformly from $\mathcal{I} = [0, 1]$.

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We consider the sequence of random variables

$$S_k^{\langle\mu\rangle} = \frac{R_\alpha(n) + 1}{n} = 1 + \frac{q_{k-1} + q_k}{n}, \qquad n = n_k^{\langle\mu\rangle}.$$

For any fixed $\mu \in [0, 1]$, we perform an asymptotic study

- for expected values: $\lim_{k\to\infty} \mathbb{E}[S_k^{\langle\mu\rangle}]$
- for distributions : $\lim_{k\to\infty} \Pr[S_k^{\langle\mu\rangle} \in J]$

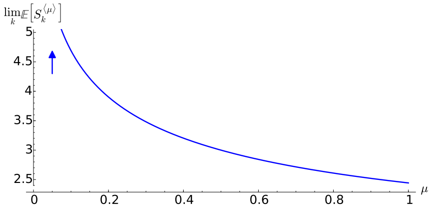
First result : Expectations

For each $\mu \in]0,1]$, the sequence of random variables $S_k^{\langle\mu\rangle}$ satisfies

$$\mathbb{E}[S_k^{\langle\mu\rangle}] = 1 + \frac{1}{\log 2}\,\frac{|\log\mu|}{1-\mu} + O\left(\frac{\varphi^{2k}}{\mu}\right) + O\left(\varphi^{k}\,\frac{|\log\mu|}{1-\mu}\right),$$

(for $k \to \infty$). Here, $\varphi = (\sqrt{5}-1)/2 \doteq 0.6180339\ldots$ and the constants of the O-terms are uniform in μ and k.

Remark: The result only holds for $\mu > 0$.



Limit of the expected value as a function of μ.

Second result : Distributions

For each $\mu \in [0, 1]$ *with* $\mu \neq 1/2$,
the sequence of random variables $S_k^{\langle \mu \rangle}$ *has a limit density*

$$s_\mu(x) = \frac{1}{\log 2\,(x-1)\;|2 - \mu - x\,(1-\mu)|}\,\mathbf{1}_{I_\mu}(x)\,.$$

Here, I_μ *is the interval with endpoints* 3 *and* $1 + 1/\mu$.

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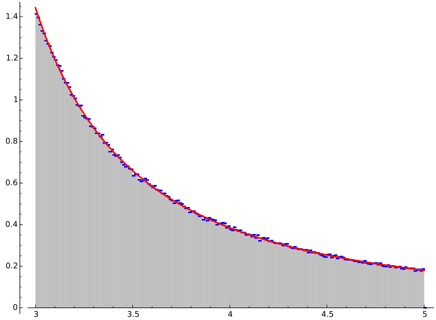
Here, I_μ *is the interval with endpoints* 3 *and* $1 + 1/\mu$.
For all $b \geq \min\{3,\ 1+\frac{1}{\mu}\}$

$$\Pr\left[S_k^{\langle\mu\rangle} \leq b\right] = \int_0^b s_\mu(x)dx + \frac{1}{b}\,O\left(\varphi^k\right)\,.$$

where the constant of the O*-term is uniform in* b *and* k.
When $|\mu - 1/2| \geq \epsilon$ *for a fixed* $\epsilon > 0$, *it is also uniform in* μ.

Limit distribution for $\mu = 1/4$

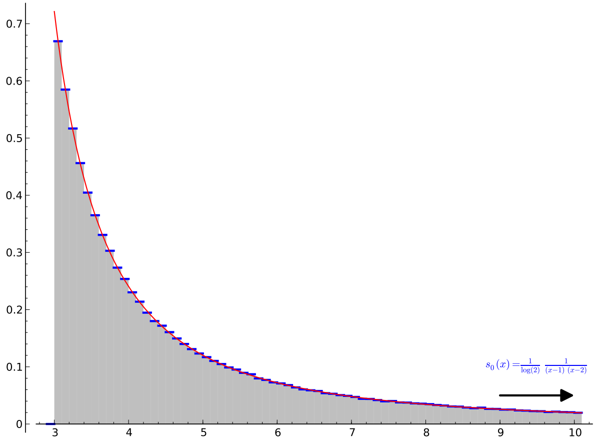
Interval	Empirical Pr	Asymptotic Pr
$[3.0, 3.0]$	0.0	0.0
$[3.0, 3.5]$	$0.\mathbf{485}237$	$0.\mathbf{485}4\ldots$
$[3.0, 4.0]$	$0.\mathbf{73}7139$	$0.\mathbf{73}69\ldots$
$[3.0, 4.5]$	$0.\mathbf{893}511$	$0.\mathbf{893}1\ldots$
$[3.0, 5.0]$	1.0	1.0



In blue, the scaled histogram for $k = 25$, bin-width $\delta = 1/10$, obtained with 10^6 samples.

In red, the graph of the limit distribution $s_{1/4}(x) = \dfrac{1}{\log 2} \dfrac{4}{(x-1)(3x-7)}$.

Limit distribution for $\mu = 0$



In blue, the scaled histogram for $\mu = 0$, $k = 25$, bin-width $\delta = 1/10$, obtained with 10^6 samples.

In red, the graph of $s_0(x) = \dfrac{1}{\log 2} \dfrac{1}{(x-1)(x-2)}$.

Four steps in the proof

i) We drop the integer part in $S_k^{\langle\mu\rangle}$ getting

$$\tilde{S}_k^{\langle\mu\rangle} = 1 + \frac{q_k + q_{k-1}}{q_{k-1} + \mu\,(q_k - q_{k-1})}\,,$$

which depends only on $\frac{q_{k-1}}{q_k}$. Indeed

$$\tilde{S}_k^{\langle\mu\rangle} = f_\mu\left(\frac{q_{k-1}}{q_k}\right), \qquad \text{with} \qquad f_\mu(x) = 1 + \frac{1+x}{x+\mu\,(1-x)}\,.$$

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ii) The expected value and the distribution of $\tilde{S}_k^{\langle\mu\rangle}$ are expressed with the k–th iterate of the Perron-Frobenius operator **H**.

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iv) Finally we return from $\tilde{S}_k^{\langle\mu\rangle}$ to $S_k^{\langle\mu\rangle}$.

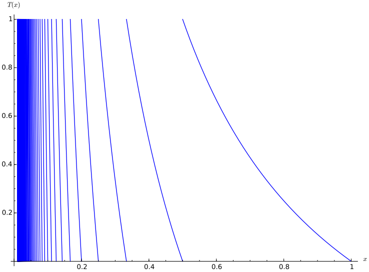
The Euclidean dynamical system

The Gauss map $T : [0,1] \to [0,1]$

$$T(x) = \left\{ \frac{1}{x} \right\} = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor .$$

The inverse branches of T are:

$$\mathcal{H} = \left\{ h_m \colon x \mapsto \frac{1}{m+x} \quad : \quad m \geq 1 \right\} .$$



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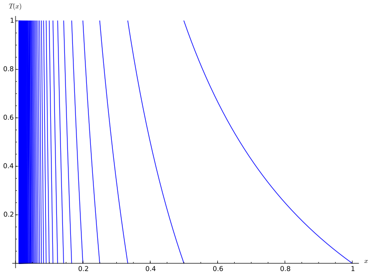

The inverse branches of T^k are:

$$\mathcal{H}^k = \{ h_{m_1, m_2, \ldots m_k} = h_{m_1} \circ h_{m_2} \circ \ldots \circ h_{m_k} \quad : \quad m_1, \ldots, m_k \geq 1 \} .$$

The LFT $h_{m_1,\dots,m_k} \in \mathcal{H}^k$ is expressed with continuants

$$h_{m_1,\dots,m_k}(x) = \cfrac{1}{m_1 + \cfrac{1}{\ddots + \cfrac{1}{m_k + x}}} = \frac{p_{k-1}\,x + p_k}{q_{k-1}\,x + q_k}\,,$$

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and satisfies the mirror property

$$h_{m_k,\dots,m_1}(x) = \cfrac{1}{m_k + \cfrac{1}{\ddots + \cfrac{1}{m_1 + x}}} = \frac{p_{k-1}\,x + q_{k-1}}{p_k\,x + q_k}\,.$$

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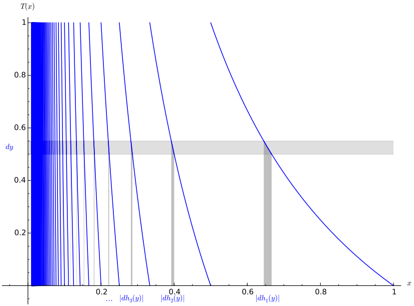

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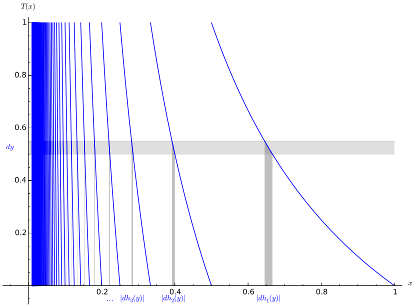

Answer: The density is

$$\mathbf{H}[g](x) = \sum_{h \in \mathcal{H}} \left| h'(x) \right| \, g\left(h(x)\right)$$

$$= \sum_{m=1}^{\infty} \frac{1}{(m+x)^2} \, g\left(\frac{1}{m+x}\right) .$$

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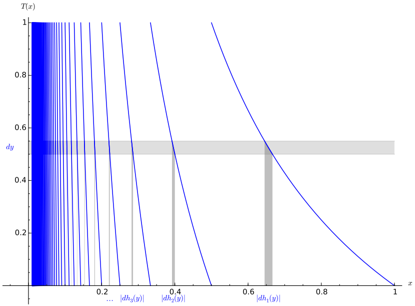

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For $k \geq 1$, the density of $T^k(\alpha)$ is given by the k-th iterate of $\mathbf{H}$

$$\mathbf{H}^k[g](x) = \sum_{h \in \mathcal{H}^k} \left|h'(x)\right| \, g\left(h(x)\right) .$$

$\mathbf{H}$ is called the Perron-Frobenius operator (or the density transform).

Evaluating at $x = 0$

$$\mathbf{H}^k[g](0) = \sum_{m_1,\dots,m_k \geq 1} \frac{1}{q_k^2}\, g\left(\frac{p_k}{q_k}\right) .$$

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As the sum is over all k-tuples, we apply the mirror property, and

$$\mathbf{H}^k[g](0) = \sum_{m_1,\ldots,m_k \geq 1} \frac{1}{q_k^2}\, g\left(\frac{q_{k-1}}{q_k}\right).$$

Expressions in terms of the operator $\mathbf{H}$.

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$$|h(0) - h(1)| = \frac{1}{q_k\,(q_k + q_{k-1})} = \frac{1}{q_k^2} \cdot \frac{1}{1 + \frac{q_{k-1}}{q_k}}$$

Expressions in terms of the operator $\mathbf{H}$.

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And $$\Pr\left[\tilde{S}_k^{\langle\mu\rangle} \in J\right] = \mathbb{E}\left[\mathbf{1}_J \circ \tilde{S}_k^{\langle\mu\rangle}\right] = \mathbf{H}^k\left[\frac{\mathbf{1}_J \circ f_\mu(x)}{1+x}\right](0)$$

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The operator **H** acts on the Banach space BV$(\mathcal{I})$ of functions of bounded variation,

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- Dominant eigenvalue (simple) : $\lambda = 1$
- Dominant eigenfunction: $\psi(x) = \dfrac{1}{\log 2} \dfrac{1}{1+x}$.
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Then, for any $g \in \mathsf{BV}(\mathcal{I})$, the asymptotic estimate holds:

$$\mathbf{H}^k[g](x) = \frac{1}{\log 2} \frac{1}{1+x} \int_0^1 g(x)dx + O\left(\varphi^{2k} \|g\|_{BV}\right) \,.$$

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With the expressions for the expectations and distributions,

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  even for $\mu = 0$ with a bounded BV-norm wrt μ.

  The limit distribution

$$\lim_{k\to\infty} \Pr\left[\tilde{S}_k^{\langle\mu\rangle} \in J\right] = \frac{1}{\log 2}\int_0^1 \frac{\mathbf{1}_J \circ f_\mu(x)}{1+x}dx,$$

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Thus the asymptotics are obtained for $\tilde{S}_k^{\langle\mu\rangle}$. We then return to $S_k^{\langle\mu\rangle}$.

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We also deal with quadratic irrationals α:
these occur for Sturmian words obtained with substitutions.

References

P. Alessandri, and V. Berthé,
Three distance theorems and combinatorics on words,
L'Enseignement Mathématique, 44, pp. 103–132, 1998.

M. Iosifescu, and C. Kraaaikamp,
Metrical Theory of Continued Fractions,
Collection Mathematics and Its Applications, Vol 547, Kluwer Academic Press, 2002.

E. Cesaratto, and B. Vallée,
Pseudo-randomness of a random Kronecker sequence. An instance of dynamical analysis,
To appear as a chapter in the book Combinatorics, Words and Symbolic Dynamics (ed. V. Berthé and M. Rigo).

J. Bourdon, B. Daireaux, and B. Vallée,
Dynamical analysis of α-Euclidean Algorithms,
Journal of Algorithms, 44, pp. 246-285, 2002.