VARIOUS POINTS OF VIEW ON SOURCES

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IN ANALYTIC INFORMATION THEORY

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APPLICATIONS to PROBABILISTIC ANALYSES
of DICTIONARY STRUCTURES

Brigitte Vallée

GREYC (CNRS and University of Caen)

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Two main objects : sources and data structures

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- Deduce consequences for the analysis of related data structures

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Plan of the talk.

– A general model of sources

– The two digital structures : trie and dst.

– Probabilistic analysis of the structures, and its two steps

– Probabilistic analysis : the combinatorial step

– Probabilistic analysis : the analytic step – Need of more regular sources.

– Analysis of data structures : the result.

Sources (I)

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a probabilistic mechanism which produces symbols from alphabet Σ,

one at each time unit.

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Compromise: Only the positive part of the history is "shown"....

The negative part of the history

– is produced

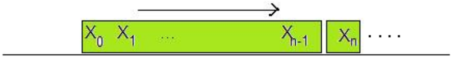
– may have an influence on the positive part

– but remains "hidden"

A source on the alphabet Σ with a positive history :

An origin for time : $t = 0$

a sequence of random variables $(X_0, X_1, \ldots, X_n, X_{n+1} \ldots)$



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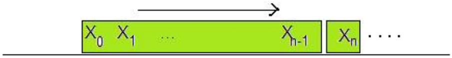

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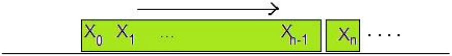

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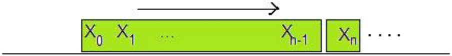

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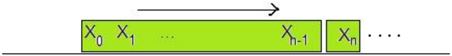

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A general source may have many, strong correlations between its symbols.

For $w \in \Sigma^\star$, $p_w :=$ probability that a word begins with the prefix w.

The set $\{p_w, \quad w \in \Sigma^\star\}$ defines the source $\mathcal{S}$.

A main analytical object related to any source:
the Dirichlet generating functions of the source

$$\Lambda(s) := \sum_{w \in \Sigma^\star} p_w^s, \qquad \Lambda^{[k]}(s) = \sum_{w \in \Sigma^k} p_w^s, \qquad \left[\Lambda = \sum_{k \geq 0} \Lambda^{[k]} \right]$$

Remark: $\Lambda^{[k]}(1) = 1$ for any k, $\qquad \Lambda(1) = \infty$.

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– they intervene in probabilistic analyses of algorithms and data structures.

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These nice expressions are due to multiplicative properties of probabilities.

And for a general source?

Does $\Lambda(s)$ admit a nice alternative expression?

A general source and its shifted sources

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For $u \in \Sigma^\star$ with $p_u \neq 0$, the source $\mathcal{S}_{(u)} = \mathcal{S}|_u$ is a shifted source

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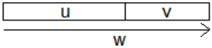
- when w is any finite prefix for which $u \leq w$.

In this case, w can be written as $w = u \cdot v$



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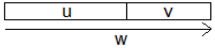
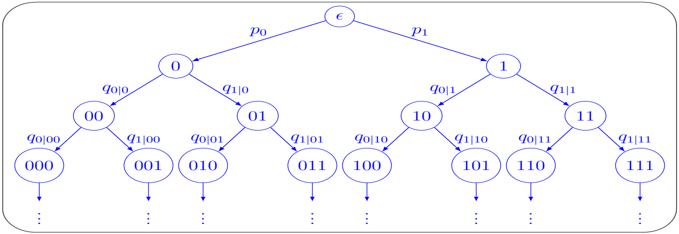


The conditional probabilities $p_{w|u} = p_{(u.v)}/p_u$ are denoted as $q_{v|u}$.

These are the fundamental probabilities of the source $S_{(u)}$.

The weighted graph of the source

- vertices = sources $\mathcal{S}_{(u)}$
- edges weighted by the probabilities $q_{v|u}$



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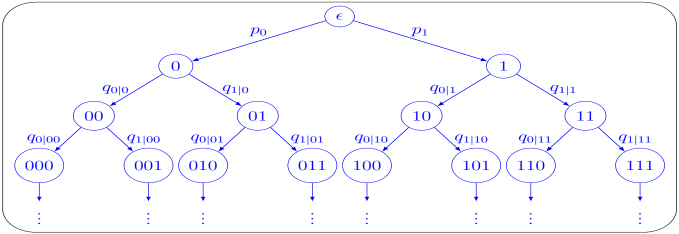

$\mathbf{P}$ = the transition matrix of the graph.

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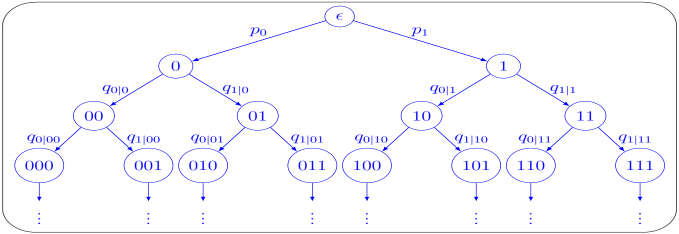

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For $s \in \mathbb{C}$, the matrix $\mathbf{P}_s$ is obtained from $\mathbf{P}$ by raising its elements to the power s

The pruned graph and the pruned matrix (I)

Sometimes, the graph (and thus the matrix) can be pruned:

With an equivalence relation on the "shifted" sources

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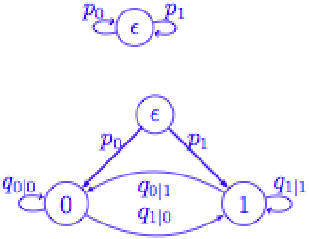
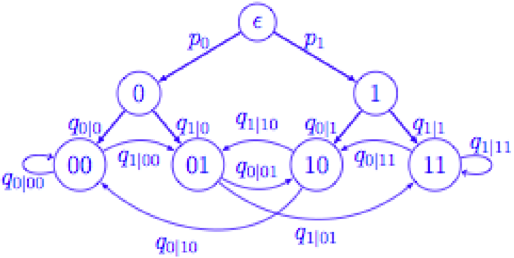
For simple sources, this provides a finite graph (a finite matrix).





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There are pruned graphs which remain infinite.

An instance of a VLMC (Variable Length Markov Chain)

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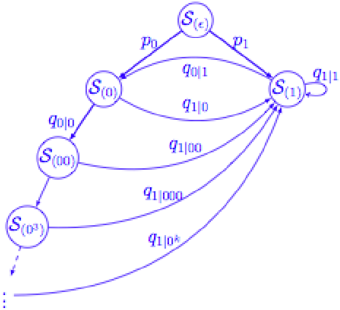

Pruned graph :

– vertices $\mathcal{S}_{(\epsilon)}, \mathcal{S}_{(1)}$ and $\mathcal{S}_{(0^k)}$ for $k > 0$
– all the edges labeled with 1
return to the source $\mathcal{S}_{(1)}$.

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A general source, with its (pruned) transition matrix $\mathbf{P}_s$,

$$\Lambda(s) = {}^t\mathbf{E} \cdot (I - \mathbf{P}_s)^{-1}[\mathbf{1}] \quad \text{with} \quad {}^t\mathbf{E} := (1, 0, 0 \ldots)$$

(II) Two data structures: trie and dst

dst : digital search tree — trie: shorthand for tree retrieval

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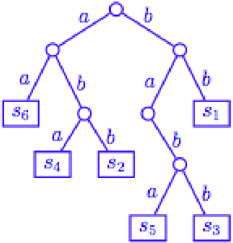
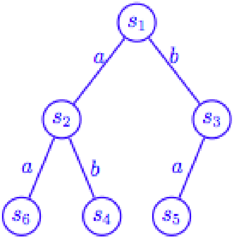
Dynamical data structures which contain words.

– Useful for sorting, and searching words.

– Important to analyze their probabilistic shape
  when built on a sequence of words emitted by a general source





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These trees direct words to subtrees according to their first symbol

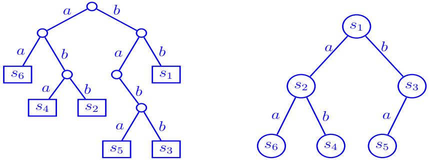
In a trie, – internal nodes do not contain data,

– the order of insertion does not intervene.

In a dst, a word is placed on the first free node.

In a trie, the word is placed when it is alone in its subtree.

$s_1{=}bbab\cdots$; $s_2{=}abbaa\cdots$ $s_3{=}babba\cdots$, $s_4{=}ababb\cdots$; $s_5{=}babab\cdots$; $s_6{=}aaaab\cdots$



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- If $|\mathcal{Y}| = 0$, $\mathrm{trie}(\mathcal{Y}) = \varnothing$.
- If $|\mathcal{Y}| = 1$, $\mathrm{trie}(\mathcal{Y}) = \boxed{Y}$
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$\mathrm{dst}(\mathcal{Y})$

- If $|\mathcal{Y}| = 0$, $\mathrm{dst}(\mathcal{Y}) = \varnothing$
- If $|\mathcal{Y}| \geq 1$, $\underline{\mathcal{Y}} := \mathcal{Y} \setminus \{\mathrm{First}(\mathcal{Y})\}$

$$\mathrm{dst}(\mathcal{Y}) = \langle \mathrm{First}(\mathcal{Y}), \mathrm{dst}(\underline{\mathcal{Y}}_{(a)}), \mathrm{dst}(\underline{\mathcal{Y}}_{(b)}) \rangle$$

Their recursive definitions

Let $\Sigma = \{a, b\}$ and a sequence $\mathcal{Y}$ of words over Σ

$\mathcal{Y}_{(\alpha)}$:= subsequence of $\mathcal{Y}$ beginning with α, the first symbol α removed.

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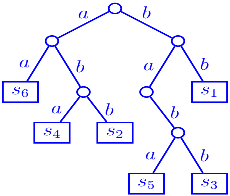

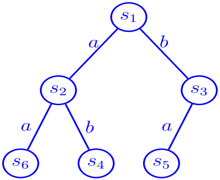

– We will use these recursive definitions to write systems of equations.

Role of the dst in the Lempel–Ziv Algorithm.

The Lempel-Ziv algorithm is a dictionary-based scheme

– it partitions a sequence into phrases of variable size

– a new phrase is the shortest substring not seen in the past as a phrase
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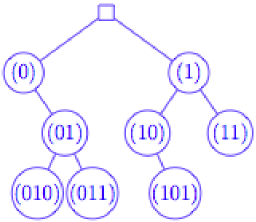

The phrases are inserted in a DST

Parameters for digital trees.

Two types of nodes in the digital structures

– nodes containing data   – or nodes containing no data

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(an internal node for the dst, an external node for the trie)

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The level of a node: the length of the path from the root to it.

The size is the number of full nodes.

The two main shape parameters:

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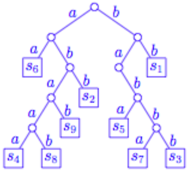
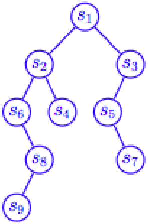


$b_{9,0} = 0,$
$b_{9,1} = 0$
$b_{9,2} = 2,$
$b_{9,3} = 1,$
$b_{9,4} = 2$
$b_{9,5} = 4$



$b_{9,0} = 1,$
$b_{9,1} = 2,$
$b_{9,2} = 3,$
$b_{9,3} = 2$
$b_{9,4} = 1$

$$D_n = (1/9)\,[2\cdot 2 + 3\cdot 1 + 4\cdot 2 + 5\cdot 4] = 3.88 \qquad D_n = (1/9)\,[1\cdot 2 + 2\cdot 3 + 3\cdot 2 + 4\cdot 1] = 2$$

(III) Probabilistic analysis of the data structures.

Probabilistic study

Input $=$ a sequence $\mathcal{X}$ of words (independently) produced by the source.

Set of inputs $=$ the set $\mathcal{M}^\star$ of such sequences $\mathcal{X}$

Aim $=$ the probabilistic shape of $\mathrm{Tree}\,(\mathcal{X})$ for $\mathcal{X} \in \mathcal{M}^\star$

Two different probabilistic models : Poisson and Bernoulli

– In the Bernoulli model, the cardinality N of $\mathcal{X}$ is fixed.

– In the Poisson model, the cardinality N follows a Poisson law of parameter z

$$\Pr[N = k] = e^{-z} \frac{z^k}{k!}.$$

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For a random variable R defined on the set $\mathcal{M}^\star$ of inputs,

there is a relation between the two expectations

$P(z)$ in the Poisson model and B_n in the Bernoulli model,

$$P(z) = e^{-z}\sum_{n\geq 0} B_n \frac{z^n}{n!}$$

Two steps in the analysis of the profile polynomial $b_N(u) := \sum_{k \geq 0} b_{N,k}\, u^k$,

Deal with the expectations of $b_N(u)$: $B_n(u)$ [Bernoulli] and $P(z,u)$ [Poisson].

(A) The first (combinatorial) step provides an exact expression for $B_n(u)$

Expectation $P(z,u)$ in the Poisson model	$\Longrightarrow$	Mellin transform $s \mapsto Z(s,u)$ of $z \mapsto P(z,u)$	$\Longrightarrow$	Binomial expression of the expectation $B_n(u)$ in the Bernoulli model

$$B_n(u) = \sum_{\ell=2}^{n} (-1)^\ell \binom{n}{\ell}\, \Delta(\ell, u), \qquad \text{with} \quad \Delta(s,u) := \frac{1}{\Gamma(-s)} Z(-s,u)$$

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(B) The second (analytic) step provides an asymptotic estimate for $B_n(u)$.

– It transforms the binomial expression into an integral expression.

– It transfers the knowledge about singularities of $s \mapsto \Delta(s,u)$ into asymptotic estimates of $B_n(u)$

– It depends on the "tameness" of $s \mapsto \Delta(s,u)$.

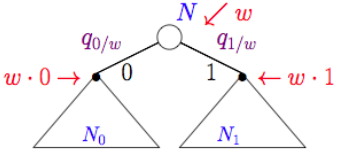
(III) Probabilistic analysis : the combinatorial step.

Profile in the Poisson model

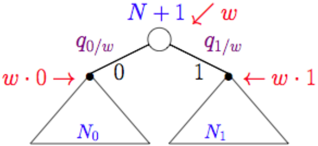
Associate with a source $\mathcal{S}$ all its shifted sources $\mathcal{S}_{(w)}$.

Profile $b^{(w)}_{N,k}$:= the number of full nodes at level k of a digital tree of size N built on the source $\mathcal{S}_{(w)}$

For a trie of size N



For a dst of size $N+1$



$$b^{(w)}_{N,k} = b^{(w\cdot 0)}_{N_0,k-1} + b^{(w\cdot 1)}_{N_1,k-1}$$

$$b^{(w)}_{N}(u) = u b^{(w\cdot 0)}_{N_0}(u) + u b^{(w\cdot 1)}_{N_1}(u)$$

$$b^{(w)}_{N+1,k} = b^{(w\cdot 0)}_{N_0,k-1} + b^{(w\cdot 1)}_{N_1,k-1}$$

$$b^{(w)}_{N+1}(u) = u b^{(w\cdot 0)}_{N_0}(u) + u b^{(w\cdot 1)}_{N_1}(u)$$

$$N = N_0 + N_1$$

The number N_j of nodes in the j-th subtree (that begin with the symbol j) follows a Poisson law of parameter $q_{j|w}\, z$

System of equations on Poisson expectations.

$$\begin{cases} P^{(w)}(z,u) & = z(1-e^{-z}) + u \sum_{i\in\Sigma} P^{(w\cdot\cdot i)}(q_{i|w}\, z, u) & \text{[for trie]} \\ P^{(w)}(z,u) & + \frac{d}{dz} P^{(w)}(z,u) = z + u \sum_{i\in\Sigma} P^{(w\cdot i)}(q_{i|w} z, u) & \text{[for dst]} \end{cases}$$

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For each type of tree,

a system of functional equations that involves in both cases

– the mapping $z \mapsto qz$ – the shift on words $w \mapsto w \cdot i$

– the derivation d/dz occurs for dst, not for tries.

$\Longrightarrow$ Analysis is more involved for dst.

The Mellin transform of the Poisson expectation.

Begin with the equations satisfied by the Poisson expectations,

$$\begin{cases} P^{(w)}(z,u) & = z(1-e^{-z}) + u \sum_{i\in\Sigma} P^{(w\cdot\cdot i)}(q_{i|w}\, z, u) \qquad \text{[for trie]} \\ P^{(w)}(z,u) & + \frac{d}{dz} P^{(w)}(z,u) = z + u \sum_{i\in\Sigma} P^{(w\cdot i)}(q_{i|w} z, u) \qquad \text{[for dst]} \end{cases}$$

Consider

– their Mellin transforms $Z^{(w)}(s,u) := \int_0^{+\infty} P^{(w)}(x,u) x^{s-1} dx$

– then $\Delta^{(w)}(s,u) := \frac{1}{\Gamma(-s)} Z^{(w)}(-s,u)$,

– then the vector $\mathbf{\Delta}(s,u)$ whose components are $\Delta^{(w)}(s,u)$.

We finally obtain a linear system for $\mathbf{\Delta}(s,u)$

which involves the transition matrix $\mathbf{P}_s$ of the source

$$\begin{cases} \mathbf{\Delta}_T(s,u) & -s\mathbf{1} & = u\,\mathbf{P}_s \mathbf{\Delta}_T(s,u) \qquad \text{[for trie]} \\ \mathbf{\Delta}_D(s,u) & -\mathbf{\Delta}_D(s+1,u) & = u\mathbf{P}_s\, \mathbf{\Delta}_D(s,u) \qquad \text{[for dst]} \end{cases}$$

with $\mathbf{1} = {}^t(1,1,1,\dots)$

The vectors $\mathbf{\Delta}(s,u)$ satisfy,

$$\begin{aligned}
\mathbf{\Delta}_T(s,u) &= s(I - u\mathbf{P}_s)^{-1}\mathbf{1} \\
\mathbf{\Delta}_D(s,u) &= (I - u\mathbf{P}_s)^{-1}\mathbf{\Delta}_D(s+1,u)
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For dst, iterate: it appears an infinite product

$$\mathbf{Q}(s,u) := (I-u\mathbf{P}_s)^{-1}\cdot\ldots(I-u\mathbf{P}_{s+k})^{-1}\ldots.$$

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Return to the initial source $\mathcal{S}$ $[\mathbf{E} := {}^t(1,0,0,\ldots)]$

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An exact expression for $\Delta(s,u)$ $\Longrightarrow$ a binomial expression for $B_n(u)$

The end of the combinatorial step.

(IV) Probabilistic analysis : the analytic step.

Return to the operator $\mathbf{P}_s$ and its quasi-inverse $(I - u\mathbf{P}_s)^{-1}$.

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The operator $\mathbf{P}_s$ operates on $L^\infty(\Sigma^\star)$ in a natural way:
$L^\infty(\Sigma^\star)$:= the Banach space of the bounded functions $X : \Sigma^\star \to \mathbb{C}$,

endowed with the sup norm.

$$Y = \mathbf{P}_s[X] \qquad \Longleftrightarrow \qquad Y(w) = \mathbf{P}_s[X](w) := \sum_{i \in \Sigma} q^s_{i|w} \, X(w \cdot i)$$

$\mathbf{P} := \mathbf{P}_1$ is stochastic, $\quad\Longrightarrow\quad$ a dominant eigenvalue equal to 1.
Need : precise information for the quasi-inverse $(I - u\mathbf{P}_s)^{-1}$

for u close to 1 and $\Re s$ close to 1.

Related to spectral properties of $\mathbf{P}_s$ on a convenient functional space....

Which functional space ?

There are two cases (for the source)

- (i) The pruned graph becomes finite
- (ii) it remains infinite.

There are two cases (for the tree) $=$ the T-case and the D-case.

For (ii) – we have to find a space where the infinite matrix $\mathbf{P}_s$ well behaves.

– there is an extra difficulty in the D-case: the infinite product

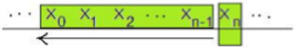
and we thus need a source with a past

Sources with a past

When the symbol X_n is emitted,

– it "looks at" (from its relative point of view) its neighbors,

– which form its reverse past $X_{n-1}, \cdots, X_1, X_0$ in this order



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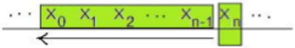

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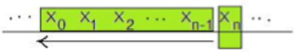

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Properties of g for "simple" sources:

Memoryless source $\iff g$ constant on each $i \cdot \Sigma^\star$, $i \in \Sigma$

Markov chains of order $1 \iff g$ constant on each $ij \cdot \Sigma^\star$, $i, j \in \Sigma$

Markov chains of order $k \iff g$ constant on each $w \cdot \Sigma^\star$, $w \in \Sigma^{k+1}$

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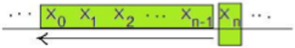

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For "good" sources: one may assume g to be continuous or even Hölder with respect to the usual "distance" δ on $\Sigma^\star$,

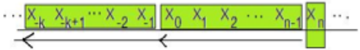
$\delta(x, y) = 2^{-\gamma(x,y)}$ where $\gamma(x, y)$ the coincidence between x and y

Sources with an infinite past.

If the source is regular enough (with a Hölder g-function for instance),

this finite reverse past can be extended to an infinite reverse past

It admits a stationary measure, and we consider the stationary source.



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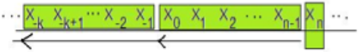

The (mirror of) transition matrix $\mathbf{P}_s$ is then extended into an operator

– which acts on the space $\mathcal{H}(\Sigma^{\mathbb{N}})$ of Hölder functions $X : \Sigma^{\mathbb{N}} \to \mathbb{C}$.

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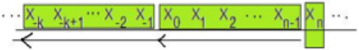

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$(I - u\mathbf{P}_s)^{-1}$ is extended to $(I - u\mathbb{H}_s)^{-1}$ which is "tame"

for $\Re s$ and u close to 1.

and the $\Delta(s, u)$ related to the two data structures

$$
\begin{aligned}
\Delta_T(s,u) &= s \ {}^t\mathbf{E}\,(I - u\mathbf{P}_s)^{-1}\,\mathbf{1}, && \text{[for trie]} \\
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\end{aligned}
$$

are also "tame", with a "tameness" of the same type.

(V) Probabilistic analysis : the result.

Main results

Consider a stationary tame source $\mathcal{S}$,

and a digital tree built on n words independently drawn from the source.

We consider a trie (type T) or a dst (type D), with $X \in \{T, D\}$

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The mean and the variance of the depth D_n satisfy

$$\mathbb{E}[D_n] = \mu \log n + \mu_X + R(n)$$

$$\mathbb{V}[D_n] = \nu \log n + \nu_X + R(n)$$

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The mean and the variance of the depth D_n satisfy

$$\mathbb{E}[D_n] = \mu \log n + \mu_X + R(n)$$
$$\mathbb{V}[D_n] = \nu \log n + \nu_X + R(n)$$

– The dominant constants μ, ν only depend on the source, not on the tree type
– The subdominant constants μ_X, ν_X depend on the source and the tree

The inequality $\mu_T > \mu_D$ holds.

– The remainder terms $R(n)$ depend on the tameness of the source.

Main results

Consider a stationary tame source $\mathcal{S}$,
and a digital tree built on n words independently drawn from the source.
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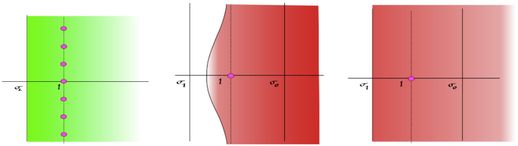
The inequality $\mu_T > \mu_D$ holds.

– The remainder terms $R(n)$ depend on the tameness of the source.

When the source S is not an unbiased memoryless source, one has $\nu \neq 0$ and the depth D_n asymptotically follows a Gaussian law

$$\frac{D_n - \mathbb{E}[D_n]}{\sqrt{\mathbb{V}[D_n]}} \xrightarrow{d} \mathcal{N}(0,1) \qquad \text{[speed of convergence } O(\log n)^{-1/2}\text{]}.$$

Precise results



$$\mathbb{E}[D_n] = \mu \log n + \mu_X + R(n), \quad \mathbb{V}[D_n] = \nu \log n + \nu_X + R(n)$$

Dominant terms	Types of tameness	Remainder terms
$\mu = -\dfrac{1}{\lambda'(1)}$	S-tame	$O(n^{-\delta})$
$\nu = \dfrac{\lambda'(1)^2 - \lambda''(1)}{\lambda'(1)^3}$	H-tame	$O\left(\exp[-(\log n)^{\rho}]\right)$
	P-tame	$\psi(n) + O(n^{-\delta})$

– $\lambda(s)$ is the dominant eigenvalue of the source $(I - \mathbb{H}_s)^{-1} \leadsto 1/(1-\lambda(s))$

– δ and ρ: related to the geometry of the tameness

– $\psi(n)$: a periodic function of $\log n$

Conclusion

Description of the interaction between the source and the data structures,

– via the $\Delta(s, u)$ functions called the mixed Dirichlet series.

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Analyses of sorting or searching algorithms when they deal with words, with the cost "number of symbols that are used for comparing words".

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Analyses of sorting or searching algorithms when they deal with words, with the cost "number of symbols that are used for comparing words".

Open question:

Is it possible to return to the analysis of the Lempel-Ziv algorithm?

What happens on the left of the vertical line $\Re s = 1$?

It is important for the analysis to deal with a region $\mathcal{R}$ where $(I - \underline{\widehat{\mathbf{P}}}_s)^{-1}$ is tame : analytic (except for $s = 1$) and of polynomial growth ($\Im s \to \infty$)

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Different possible regions $\mathcal{R}$ where $(I - \underline{\widehat{\mathbf{P}}}_s)^{-1}$ is tame.

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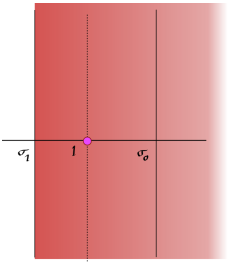
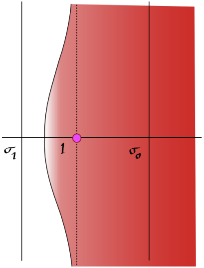


Situation 1

Vertical strip

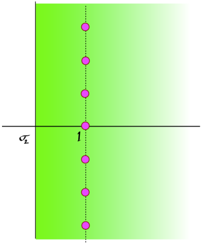
$1 - \sigma \leq a$



Situation 2

Hyperbolic region

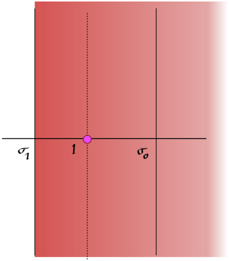
$1 - \sigma \leq t^{-\alpha}$



Situation 3

Vertical strip with holes

Possible tameness regions for a simple source



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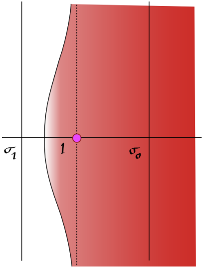
Vertical strip



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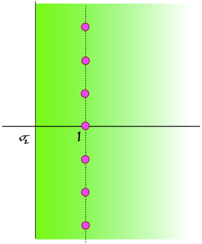
Hyperbolic region



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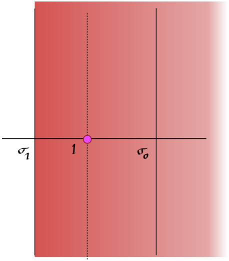

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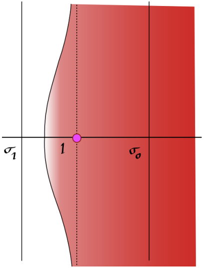
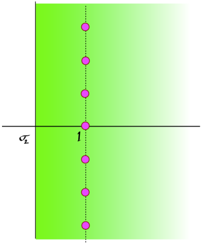


Situation 2

Hyperbolic region

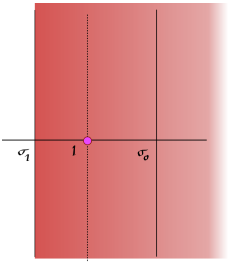


Situation 3

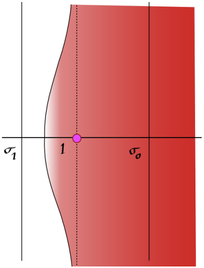
Vertical strip with holes

For which simple sources do these different situations occur?

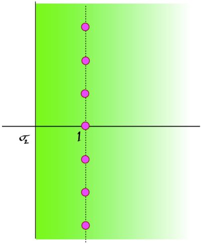
Possible tameness regions for a simple source



Situation 1
Vertical strip



Situation 2
Hyperbolic region



Situation 3
Vertical strip with holes

For which simple sources do these different situations occur?

For memoryless sources relative to probabilities $(p_1, p_2, \ldots, p_r)$

– S1 is impossible

– S3 occurs when all the ratios $\log p_i / \log p_j$ are rational

– S2 occurs if there exists a ratio $\log p_i / \log p_j$

which is "diophantine" [badly approximable by rationals]

For which Lipschitz, stationary, smooth sources do these different situations occur?

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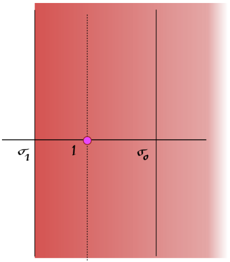
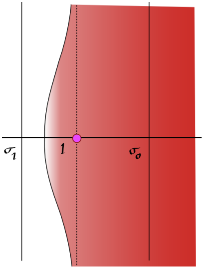


Situation 1

Vertical strip

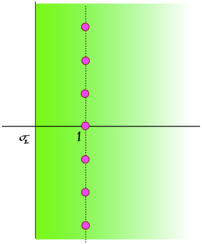
Geometric condition



Situation 2

Hyperbolic region

Arithmetic condition

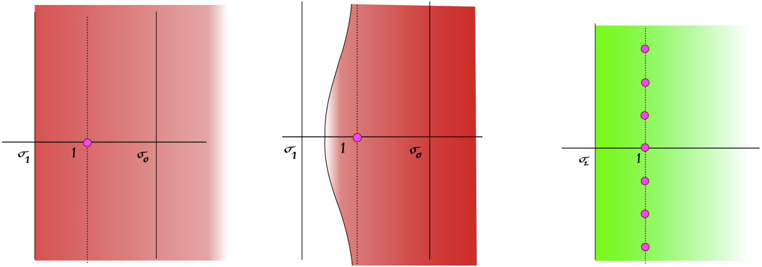


Situation 3

Vertical strip with holes

Periodicity condition

For which Lipschitz, stationary, smooth sources do these different situations occur?



Situation 1	Situation 2	Situation 3
Vertical strip	Hyperbolic region	Vertical strip with holes
Geometric condition	Arithmetic condition	Periodicity condition

– S1: When ? Find some equivalent of the UNI Condition

‘the branches are not too often of the same shape” (??)

– S3: only when the source is conjugated to a simple source.

– S2: when the following condition [DIOP] holds

“there exists two cycles $\mathcal{C}_i$ and $\mathcal{C}_j$

for which the ratio $\log p(\mathcal{C}_i)/\log p(\mathcal{C}_j)$ is “diophantine”